OXFORD IB DIPLOMA PROGRAMME

MIXED REVIEW

MATHEMATICS:
ANALYSIS AND APPROACHES

HIGHER LEVEL

COURSE COMPANION

ENHANCED ONLINE

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OXFORD

2 Representing relationships: functions

1 Simplify the difference of the binomial coefficients $\binom{n}{3} - \binom{2n}{2}$, $n \geq 3$, and hence solve the inequality $\binom{n}{3} - \binom{2n}{2} > 32n$.

2 A sum of \$5000 is invested at a compound interest rate of 6.3% per year.

a Determine the value of the investment after 5 years.

b Determine how many years it will take for the investment to exceed \$10,000.

3 A geometric sequence has first term 2 and a common ratio of 1.05. Find the smallest term that is greater than 500.

4 The sum to infinity of a geometric series is 32. The sum of the first four terms is 30, and all terms are positive. Find the difference between the sum to infinity and the sum of the first eight terms.

5 If $f(x) = (x+2)^2 - 3$ and $g(x) = ax + b$,

a find *a, a*>0, and *b* if $f(g(x)) = 4x^2 + 6x - \frac{3}{4}$.

b If $h(x) = 5x + 2$ and $k(x) = cx^2 - x + 2$, find *c* such that $h(k(x)) = 0$ has equal roots.

6 It is given that $f(x) = \sqrt{\frac{1}{x^2} - 2}$.

a Find all *x* for which *f* is real and finite.

b Find the range of *f*.

7 $f(x) = \frac{2x-1}{x+2}$ is a one-to-one function whose domain is $A = \{x | x > 0\}$.

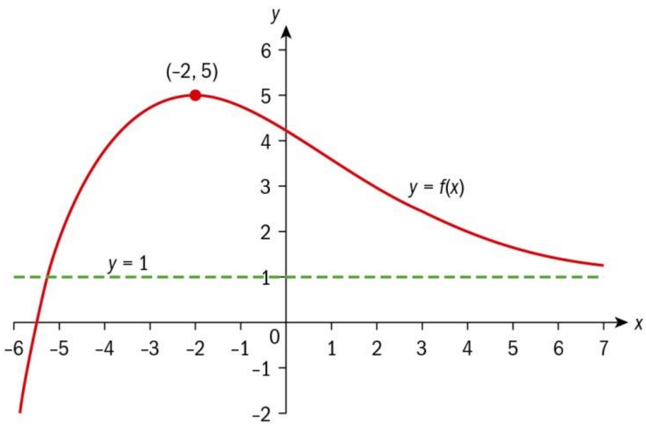
a State the range, *B*, of *f*.

b Find $f^{-1}(x)$, $x \in B$.

8 If $f(x) = 2x - 1$ and $g(x) = \frac{x}{x+1}$, $x \neq -1$, find the values of x for which $f(g(x)) \leq g(f(x))$.

9 Below is the graph of f(x) whose maximum is (-2,5) and its horizontal asymptote is $y=1$. State the maximum point and equation of the asymptote after each transformation on the original curve.

a $f(x+3)-2$ **b** $-5f(2x)$



Exam-style questions

10 A quadratic sequence has an nth term of $n^2 - 4n + 21$.

a Find the 6th term of the sequence. (1)

b Prove that every term in the sequence is positive. (4)

11 **a** Find the domain of x for which the geometric series $1+\left(\frac{2x}{3}\right)+\left(\frac{2x}{3}\right)^2+\ldots$ has a finite sum. (3)

b Find the sum of the series if $x = 1.2$. (2)

12 Given that $f(x)=(1-\frac{1}{2}x)^8$, find the coefficient of x^3. (4)

13 Consider the product $P(n)=\left(1-\frac{1}{2^2}\right)\left(1-\frac{1}{3^2}\right)\left(1-\frac{1}{4^2}\right)\ldots\left(1-\frac{1}{n^2}\right)$, $n\in\mathbb{Z}^+$, $n>1$.

a Given that $P(2)=\frac{a}{4};P(3)=\frac{b}{6};P(4)=\frac{c}{8}$, find a, b and c. (3)

b Suggest an equivalent expression for $P(n)$ as a rational number in terms of n. (2)

c By using mathematical induction, prove that the product $P(n)=\left(1-\frac{1}{2^2}\right)\left(1-\frac{1}{3^2}\right)\left(1-\frac{1}{4^2}\right)\ldots\left(1-\frac{1}{n^2}\right)$ and your expression from part (b) are equivalent. (7)

14 **a** Prove, by the method of mathematical induction, that the discrete function $f(n)=1^2+2^2+3^2+\ldots+n^2$ is equal to $\frac{1}{6}n(n+1)(2n+1)$ for $n\in\mathbb{Z}^+$. (7)

b Hence, find an expression for $3^2+6^2+9^2+\ldots+(3n)^2$. (2)

c Given that $S_1=1^2+4^2+7^2+\ldots+(3n-2)^2$ and $S_2=2^2+5^2+8^2+\ldots+(3n-1)^2$, prove that

i $S_2+S_1=6n^3-n$ (3)

ii $S_2-S_1=3n^2$ (4)

d Hence, or otherwise, find S_1 and S_2 in terms of n. (4)

Answers

1 $\binom{n}{3} = \frac{n(n-1)(n-2)}{6}$, $\binom{2n}{2} = n(2n-1)$, so $\binom{n}{3} - \binom{2n}{2} = \frac{1}{6}n(n^2 - 15n + 8)$ and

$\frac{1}{6}n(n^2 - 15n + 8) > 32n \Rightarrow n^2 - 15n + 8 > 192 \Rightarrow (n-23)(8+n) > 0 \Rightarrow -8 < n < 0, n > 23$.

2 **a** \$6790

b 12 years

3 521

4 $\frac{1}{8}$

5 **a** a=2, b=-1/2

b $c = \frac{5}{48}$

6 **a** $-\frac{1}{\sqrt{2}} \le x \le \frac{1}{\sqrt{2}}$

b $y \ge 0$

7 **a** $B = \left]-\frac{1}{2}, 2\right[$

b $f^{-1}(x) = \frac{1+2x}{2-x}$

8 $-1<x<0$ or $x \ge 1/3$

9 **a** (-5,3); $y=-1$

b (-1,-25); $y=-5$

10 a $6^2 - 4(6) + 21 = 33$ A1

b Completing the square, $n^2 - 4n + 21 = (n-2)^2 + 17$ M1A2

Since $(n-2)^2 \ge 0$ for all integer n, $(n-2)^2 + 17 > 0$ for all integer n. R1

11 a The series has a finite sum when $|r|<1$.

$\left|\frac{2x}{3}\right| < 1 \Rightarrow -1.5 < x < 1.5, x \ne 0$ R1A1A1

b $s_\infty = \frac{a}{1-r} = \frac{1}{1-(\frac{2\times 1.2}{3})} = 5$ (M1)A1

12 The coefficient of x^3 is ${}^8C_3\left(-\frac{1}{2}\right)^3(1)^5$ M1A1A1

$= -7$ A1

13 a $P(2) = \frac{a}{4} = 1 - \frac{1}{2^2} = \frac{3}{4} \Rightarrow a = 3$; A1

$P(3) = \frac{b}{6} = \left(1 - \frac{1}{2^2}\right)\left(1 - \frac{1}{3^2}\right) = \frac{4}{6} \Rightarrow b = 4$ A1

$P(4) = \frac{c}{8} = \left(1 - \frac{1}{2^2}\right)\left(1 - \frac{1}{3^2}\right)\left(1 - \frac{1}{4^2}\right) = \frac{5}{8} \Rightarrow c = 5$ A1

b $P(n) = \frac{n+1}{2n}$, $n \geq 2$ A2

c Prove that $P(n) = \left(1 - \frac{1}{2^2}\right)\left(1 - \frac{1}{3^2}\right)\left(1 - \frac{1}{4^2}\right)\ldots\left(1 - \frac{1}{n^2}\right) = \frac{n+1}{2n}$, $n \geq 2$

Show that P(2) is true:

LHS $= 1 - \frac{1}{2^2} = \frac{3}{4}$; RHS$= \frac{3}{4}$, hence P(2) is true. M1

Assume $P(k)$ is true, i.e. that

$P(k) = \left(1 - \frac{1}{2^2}\right)\left(1 - \frac{1}{3^2}\right)\left(1 - \frac{1}{4^2}\right)\ldots\left(1 - \frac{1}{k^2}\right) = \frac{k+1}{2k}$, $k \in \mathbb{Z}^+$, $k \geq 2$ M1

Required to prove that $P(k+1)$ is true, i.e. that

$\left(\frac{k+1}{2k}\right)\left(1 - \frac{1}{(k+1)^2}\right) = \frac{k+2}{2(k+1)}$ M1

LHS:

$\left(\frac{k+1}{2k}\right)\left(1 - \frac{1}{(k+1)^2}\right)$

$= \frac{k+1}{2k} - \frac{1}{2k(k+1)}$ A1

$= \frac{(k+1)^2 - 1}{2k(k+1)}$ A1

$= \frac{k+2}{2(k+1)}$ A1

=RHS

Since $P(2)$ was shown to be true, and it was proved that if $P(k)$ is true, then $P(k+1)$ is also true, it follows by the principle of mathematical induction that $P(n)$ is true,

$n \in \mathbb{Z}^+$, $n \geq 2$. R1

14 a $P(n): 1^2 + 2^2 + 3^2 + \ldots + n^2 = \frac{n(n+1)(2n+1)}{6}$

Show $P(1)$ is true:

LHS: $1^2 = 1$

RHS: $\frac{1}{6} \times 1 \times 2 \times 3 = \frac{6}{6} = 1$ M1

Hence $P(1)$ is true.

Assume $P(n)$ is true for $n = k$, $k \in \mathbb{Z}^+$, i.e. that

$P(k): 1^2 + 2^2 + 3^2 + \ldots + k^2 = \frac{k(k+1)(2k+1)}{6}$ M1

Required to prove that $P(n)$ is true for $n = k + 1$, i.e. that

$1^2 + 2^2 + 3^2 + \ldots + k^2 + (k+1)^2 = \frac{(k+1)(k+2)(2k+3)}{6}$ M1

LHS: $1^2 + 2^2 + 3^2 + \ldots + k^2 + (k+1)^2 = P(k) + (k+1)^2$ A1

$= \frac{k(k+1)(2k+1)}{6} + (k+1)^2$ by the assumed result for $P(k)$

$= \frac{k(k+1)(2k+1) + 6(k+1)^2}{6}$

$= \frac{(k+1)\left[k(2k+1) + 6(k+1)\right]}{6}$ M1

$= \frac{(k+1)(2k^2 + 7k + 6)}{6}$ A1

$= \frac{1}{6}(k+1)(k+2)(2k+3)$ =RHS

Since $P(1)$ was shown to be true, and it was also proved that if $P(k)$ is true, then $P(k+1)$ is also true, it follows by the principle of mathematical induction that $P(n)$ is also true for $n \in \mathbb{Z}^+$. R1

b $3^2 + 6^2 + 9^2 + \ldots + (3n)^2 = 3^2(1 + 2^2 + 3^2 + \ldots + n^2)$ M1

$= 9\left(\frac{1}{6}n(n+1)(2n+1)\right)$ from part **a** R1

$= \frac{3}{2}(n(n+1)(2n+1))$ AG

c **i** $S_2 + S_1 = 1^2 + 2^2 + 4^2 + 5^2 + \ldots + (3n+2)^2 + (3n-1)^2$

$= 1^2 + 2^2 + 3^2 + \ldots + (3n)^2 - (3^2 + 6^2 + 9^2 + \ldots + (3n)^2)$ A1

$= \frac{1}{6}(3n)(3n+1)(6n+1) - \frac{3}{2}n(n+1)(2n+1)$ (by parts **a** and **b)**

$= \frac{n}{2}(3n+1)(6n+1) - \frac{3n}{2}(n+1)(2n+1)$ M1

$= \frac{n}{2}\left[((3n+1)(6n+1)) - 3((n+1)(2n+1))\right]$

$= \frac{n}{2}\left((18n^2+9n+1) - 3(2n^2+3n+1)\right)$

$= \frac{n}{2}(12n^2-2)$ A1

$= 6n^3 - n$ AG

ii $S_2 - S_1 = (2^2-1^2)+(5^2-4^2)+(8^2-7^2)+\ldots+\left((3n-1)^2-(3n-2)^2\right)$ A1

$= 3+9+15+21+\ldots+(6n-3)$

$= 3(\underbrace{1+3+5+7+\ldots+(2n-1)}_{\text{Arithmetic series}})$ A1

$S_{2n-1} = \frac{n}{2}(u_1+u_{2n-1}) = \frac{n}{2}(1+2n-1) = n^2$ M1A1

$\therefore S_2 - S_1 = 3n^2$ AG

d $\left.\begin{cases} S_2 + S_1 = 6n^3 - n \\ S_2 - S_1 = 3n^2 \end{cases}\right\} \Rightarrow 2S_2 = 6n^3+3n^2-n$ (M1)A1

$S_2 = \frac{n}{2}\left(6n^2+3n-1\right)$ A1

$\therefore S_1 = 6n^3 - n - \frac{n}{2}(6n^2+3n-1)$

$= 3n^3 - \frac{3n^2}{2} - \frac{n}{2}$ A1

3 Expanding the number system: complex numbers

1 Given that $2+3\mathrm{i}$ and $4-5\mathrm{i}$ are the first and the third term of an arithmetic sequence respectively. What is the sixth term of the sequence?

2 Find the domain and the range of the function $f(x)=2-\sqrt{3-5x}$.

3 Find the sum $1-2+3-4+5+6+7-8+\ldots+63-64$.

4 The real functions $f_1(x)=2x+1, f_2(x)$ and $f_3(x)=1-x$ form a geometric sequence.

a Find $y=f_2(x)$ given that the range of the function is the set of non-negative real numbers.

b What is the natural domain of the function $y=f_2(x)$?

5 Prove by mathematical induction: $\frac{1}{\ln 2 \ln 4}+\frac{1}{\ln 4 \ln 8}+\ldots+\frac{1}{\ln 2^n \ln 2^{n+1}}=\frac{n}{(n+1)(\ln 2)^2}$, n is a positive integer.

6 The polynomial $f(x)=4x^4+ax^3+bx^2+cx+9$, where $a,b,c\in\mathbb{R}$. Given that $f(3)=f(-3)$ and $f(5)=f(-5)$,

a Show that f is even.

b Find the possible values of b such that $f(x)=(px^2+q)^2, p,q\in\mathbb{R}$.

7 Solve the inequality: $5\cdot 9^x-2^3\cdot 15^x+3\cdot 25^x>0$

8 Given that $z^2-z+1=0$ evaluate $(3z^2-5z)(5z^2-3z)$.

9 The polynomial $f(x)=x^3-6x^2+px+10, p\in\mathbb{R}$. x_1, x_2 and x_3 are zeroes that form an arithmetic sequence.

a Show that $x_1+x_2+x_3=6$.

b Find $x_1\cdot x_2\cdot x_3$.

c Hence find all the zeroes.

d Calculate the value of p.

10 The quadratic function $f(x)=ax^2+bx+c, a,b,c\in\mathbb{R}, a\neq 0$ are passing through the points A(-3,4), B(-1,-6) and C(3,22).

a Write the three equations and find the parameters a, b and c.

b Given that $z = 5 - 6\text{i}$, calculate $f(z)$.

c Show that $f(z \cdot z^*) < f(z) \cdot f(z^*)$.

d Show that the polynomial of the form $f(x) = x^2 + (2k-1)x + (2m-1), k, m \in \mathbb{Z}$ has no rational zeroes.

Exam-style questions

11 **a** Find the remainder when the cubic polynomial function $f(x) = x^3 - 4x^2 - 3x + 18$ is divided by the linear function $g(x) = x - 2$. (1)

The function h is defined as $h(x) = f(x) - 4$.

b By using your result from part (a), find the roots of the equation $h(x) = 0$. (6)

12 A sum of $100 is invested.

a If the interest is compounded annually at a rate of 5% per annum find, to the nearest dollar, the value of the investment ($P) after 20 years. (2)

The interest is now compounded monthly at a rate of $\frac{5}{12}\%$.

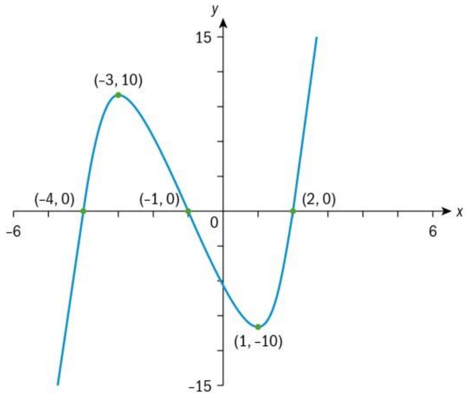
b Write an inequality that will enable you to find the minimum number of months n for the value of the investment to exceed $P. (2)

c Find the value of n. (2)

13 Given that z_1 and z_2 are complex numbers, solve the simultaneous equations $z_1 + 3z_2 = 7$ and $z_1 + \text{i}z_2 = 4 + 4\text{i}$, giving your answers in the form $z = a + b\text{i}$ where $a, b \in \mathbb{R}$. (5)

14 Given that $f(x) = (1 + 2x - 3x^2)^5$, expand f in powers of x up to and including the term in x^3. (6)

15 The graph of $y = f(x)$ is shown below.



On two separate graphs, sketch the graph of f, and use it to sketch the graphs of each of the following two functions. Indicate clearly on each sketch any asymptotes (use dotted lines) and label any axes intercepts with their coordinates. You should also label the coordinates of any local maxima and local minima.

a $\frac{1}{f(x)}$ (5)

b $|f(-x)|$ (8)

Answers

1 $7-17i$

2 $D: x \in \left]-\infty, \frac{3}{5}\right]$ $R: y \in \left]-\infty, 2\right]$

3 1798

4 a $f_2(x) = \sqrt{1+x-2x^2}$

b [-0.5,1]

5 The base case $n=1$ is $\frac{1}{(1+1)(\ln 2)^2} = \frac{1}{\ln 2 \ln 4}$ which is true. We assume that

$\frac{1}{\ln 2 \ln 4} + \frac{1}{\ln 4 \ln 8} + \ldots + \frac{1}{\ln 2^n \ln 2^{n+1}} = \frac{n}{(n+1)(\ln 2)^2}$ is true for some $n \in \mathbb{N}$.

Then consider the $n+1$ case and use the assumed result

$$\frac{1}{\ln 2 \ln 4} + \frac{1}{\ln 4 \ln 8} + \ldots + \frac{1}{\ln 2^n \ln 2^{n+1}} + \frac{1}{\ln 2^{n+1} \ln 2^{n+2}} = \frac{n}{(n+1)(\ln 2)^2} + \frac{1}{\ln 2^{n+1} \ln 2^{n+2}}.$$

Therefore $\frac{n}{(n+1)(\ln 2)^2} + \frac{1}{(n+1)(n+2)\ln 2 \ln 2} = \frac{1}{(n+1)(\ln 2)^2}\left(n + \frac{1}{n+2}\right) = \frac{n+1}{(n+2)(\ln 2)^2}$.

We conclude that $\frac{1}{\ln 2 \ln 4} + \frac{1}{\ln 4 \ln 8} + \ldots + \frac{1}{\ln 2^n \ln 2^{n+1}} = \frac{n}{(n+1)(\ln 2)^2}$ is true for all $n \in \mathbb{N}$.

6 a $a = c = 0$

b $b = \pm 12$

7]0,1[

8 19

9 a The polynomial can be factorized as $f(x) = (x-x_1)(x-x_2)(x-x_3)$. Expanding this and comparing coefficients we find that the coefficient next to x^2 is $-(x_1+x_2+x_3) = -6$ and so $x_1+x_2+x_3 = 6$.

b -10

c -1, 2, 5

d 3

10 a a=2, b=3, c=-5

b -12-138i

c 7620 < 19188

d Suppose we have a rational root $x = p/q,\ p, q \in \mathbb{Z}$. By the rational root test, p divides $(2m-1)$ and hence is an odd integer, and $q = 1$. Then $\left(\frac{p}{q}\right)^2 + (2k-1)\left(\frac{p}{q}\right) + (2m-1) = 0$

$p^2 + (2k-1)q + (2m-1) = 0$. However each term in this sum is odd, and odd number of odd integers cannot sum to zero. Therefore we reach a contradiction, and x cannot be rational.

11 a $f(2) = 2^3 - 4(2)^2 - 3(2) + 18 = 4$ A1

b Since 4 is the remainder when *f* is divided by *g*, so $(x - 2)$ must be a factor of $h(x) = x^3 - 4x^2 - 3x + 14$ R1

Therefore, $h(x) = (x-2)(x^2 - 2x - 7)$ A1

Using the quadratic formula on $(x^2 - 2x - 7)$,

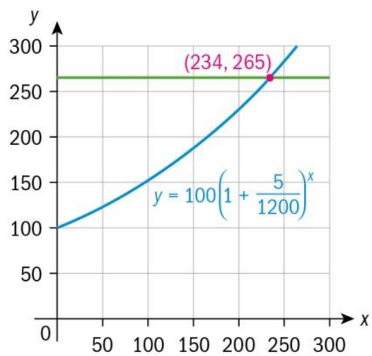
$x = \frac{2 \pm \sqrt{4+28}}{2} = \frac{2 \pm \sqrt{32}}{2} = \frac{2 \pm 4\sqrt{2}}{2} = 1 \pm 2\sqrt{2}$ M1

The roots are 2 and $1 \pm 2\sqrt{2}$. A3

12 a $P = 100(1+0.05)^{20} = \$265$ M1A1

b $100\left(1 + \frac{0.05}{12}\right)^n > 265$ M1A1

c



(solving using GDC) (M1)

$n = 235$ A1

13 $\left.\begin{aligned} 2z_1 + 3z_2 &= 7 \\ 2z_1 + 2iz_2 &= 8 + 8i \end{aligned}\right\}$ (solving simultaneously) M1

$z_2(2i - 3) = 1 + 8i$ M1

$z_2 = \frac{1+8i}{2i-3} = 1 - 2i$ M1A1

$z_1 = \dfrac{7-3(1-2i)}{2} = 2+3i$ A1

14 Using the binomial theorem $(a+b)^n = {}^nC_0 a^n b^0 + {}^nC_1 a^{n-1} b + {}^nC_2 a^{n-2} b^2 + \ldots$

with $a=1$ and $b = 2x - 3x^2$, (M1)A1

$f(x) = (1+(2x-3x^2))^5$

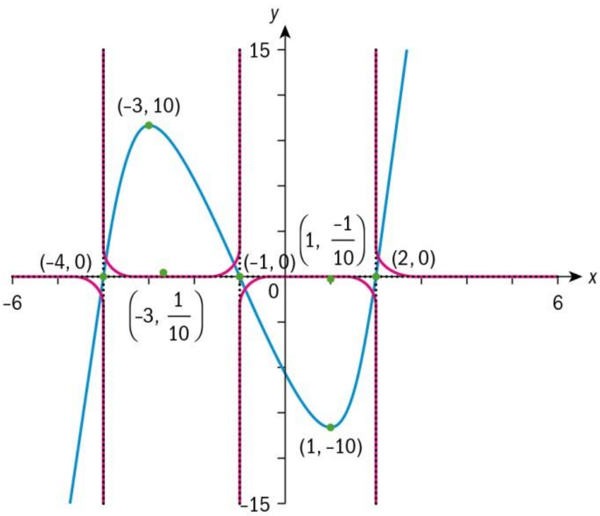
$= 1 + {}^5C_1(2x-3x^2) + {}^5C_2(2x-3x^2)^2 + {}^5C_3(2x-3x^2)^3 + \ldots$ M1

$= 1 + 10x - 15x^2 + 10(4x^2 - 12x^3 + 9x^4) + 10(8x^3 \ldots) + \ldots$ A1

$= 1 + 10x - 15x^2 + 40x^2 - 120x^3 + 80x^3 + \ldots$

$= 1 + 10x + 25x^2 - 40x^3 + \ldots$ A2

15 a $\dfrac{1}{f(x)}$



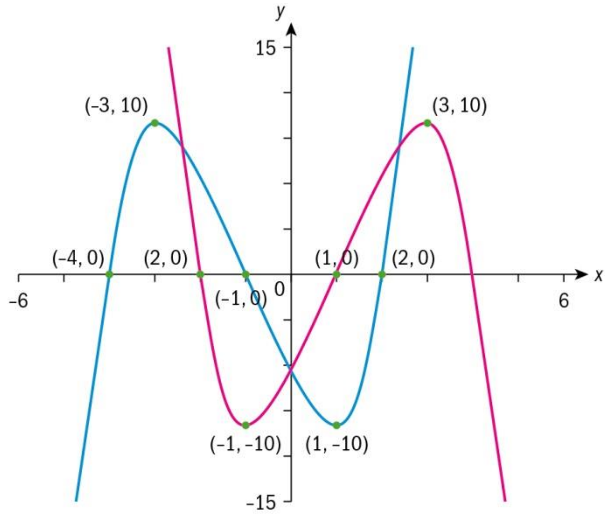
General shape A1

Asymptotes (A1 for the 3 vertical asymptotes and A1 for the horizontal asymptote; A2)

Local maximum A1

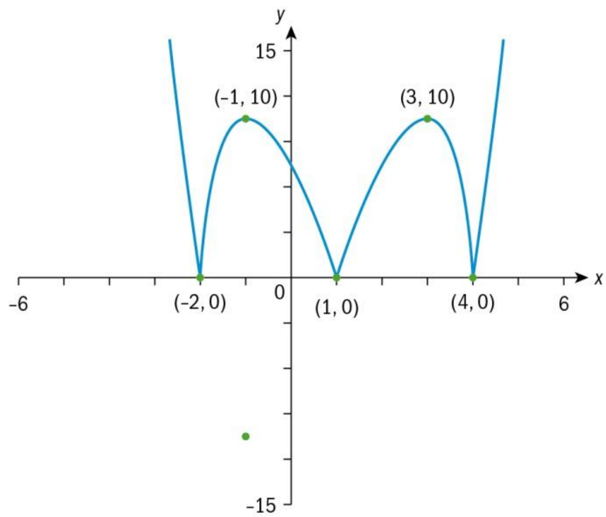
Local minimum A1

b An initial sketch of $y = f(-x)$ may be used, as shown below.



(M1)(A1)

Sketching the modulus of $y = f(-x)$ gives $y = |f(-x)|$



General shape A1

Intercepts A3

Maximum/minimum A2

4 Measuring change: differentiation

1 Prove using mathematical induction: $\frac{d(x^n)}{dx} = nx^{n-1}, x \in N$.

2 Let $f(r) = r^3 + r^2 - 5r - 2$.

a Show that r=2 is a solution of the equation $f(x)$=0.

b Find the values of p and q such that $f(r) = (r-2)(r^2 + pr + q)$.

c Hence, find the other roots of $f(r) = 0$.

d An arithmetic sequence has r as its common difference, and a geometric sequence has r as its common ratio. For both sequences, a_1=1.

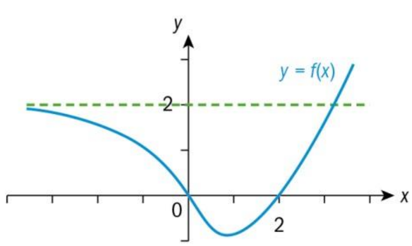
i Write down the first four terms of both sequences, in terms of r.

ii If the sum of the third and fourth terms of the arithmetic sequence is equal to the sum of the third and four terms of the geometric sequence, find the three possible values of r.

iii Find the value of r from **ii** for which S_{∞} exists.

iv For the value of r that you found in **iii**, find the sum of the first 20 terms of the arithmetic sequence, writing your answer in the form $s + t\sqrt{v}, s, t, v \in \mathbb{Z}$. .

3 The diagram shows the graph of y=f(x), such that $\lim_{x \to -\infty} f(x) = 2$.



a Sketch the graph of $y = \frac{1}{f(x)}$.

b Sketch the graph of $y = xf(x)$.

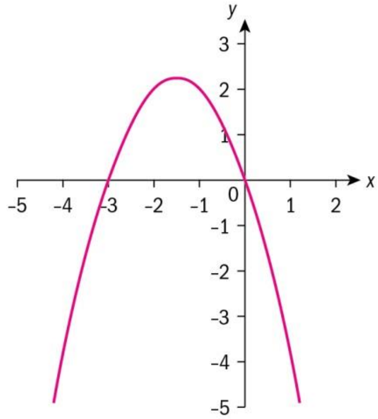
4 The graph of $f(x) = 8x^3 + rx^2 + sx + t$ has a gradient of zero at two distinct points A and B.

a Show that $r^2 > 24s$.

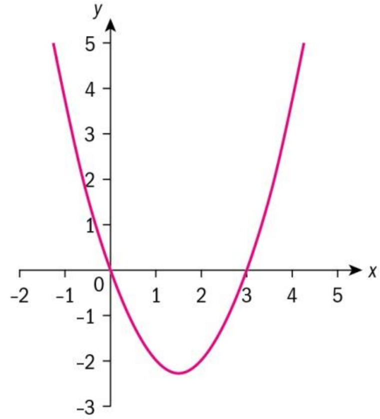
b The coordinates of A and B are (0.5,-12) and (-1.5,20) respectively. Find the values of r, s and t.

Exam-style questions

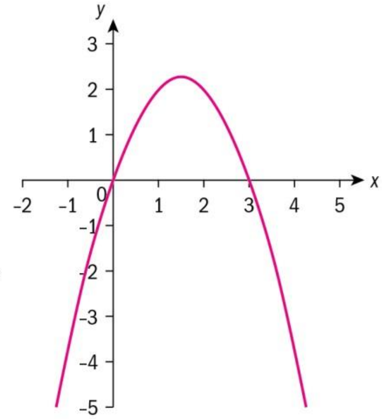
5 **a** Match each quadratic graph with the graph of its derivative. (3)



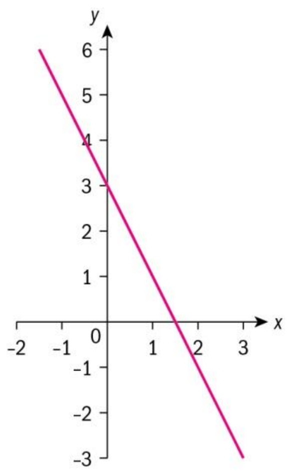
(A)



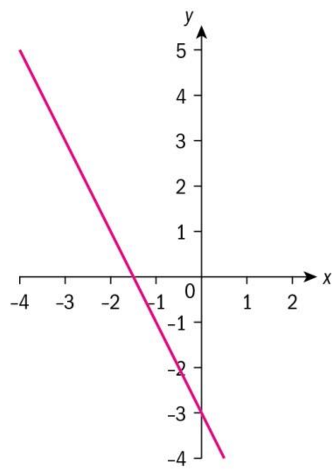
(B)



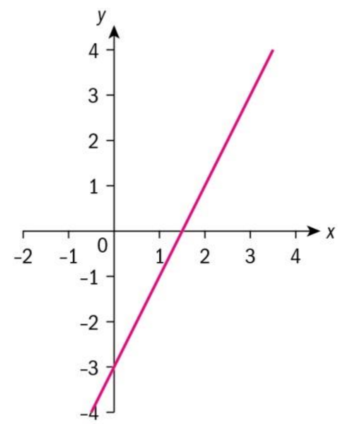
(C)



(1)



(2)

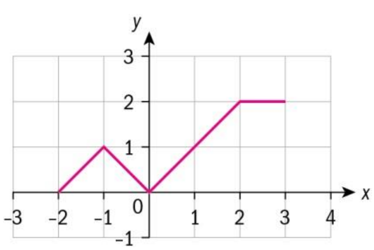


(3)

b State the zeros of the quadratic function shown in graph (A). (1)

c Write down the gradient of the tangent to graph (B) at $x = 0$. (1)

6 Consider the function f defined by the following graph.



Find

a the domain of f (1)

b the zeros of f (1)

c the range of f (1)

d the intervals where the derivative of f is positive (2)

e a point on the graph where the tangent is horizontal (1)

f the points on the domain where the derivative is not defined. (1)

7 Consider the graph of the function defined by $f(x) = x^3 + 2x^2 - 4x - 5$.

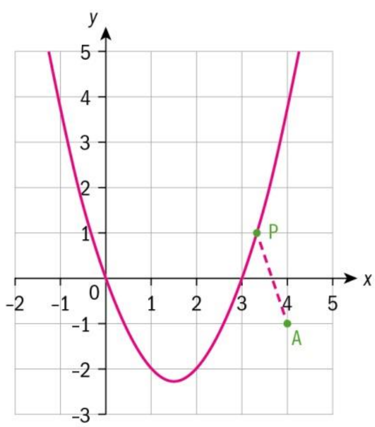
a Show that $f'(x) = (3x-2)(x+2)$. (2)

b Find the equation of the normal to the graph of f at any points where the tangent to the graph at this point is horizontal. (3)

c Find the equation of the normal to the curve at $x = 2$. (6)

d Hence, find the coordinates of the points where the normal found in part **d** meets the normals found in part **b**. (3)

8 A quadratic function is defined by $f(x) = x^2 - 3x$. Point A has coordinates $A(4, -1)$. Let P be a point on the graph of f.



a State the coordinates of P in terms of x. (1)

b Determine the coordinates of P such that the perimeter of the triangle OAP is minimum. (6)

9 The functions $f(x)$ and $g(x)$ are defined by $f(x) = \frac{x-1}{x}$ and $g(x) = \frac{x+2}{x-1}$.

a Show that g is a self-inverse. (3)

b Find an expression for $(f \circ g)'(x)$. (5)

10 Consider the function $f(x) = \sqrt{\frac{2x}{4-x}}$.

a Justify that the largest possible domain of f is $0 \le x < 4$. (3)

b Show that $f'(x) = \frac{4}{(4-x)^2}\sqrt{\frac{4-x}{2x}}$ for all $0 < x < 4$. (4)

c Justify that the function is increasing over the entire domain. (4)

d Hence find the range of f . (3)

Answers

1 The base case $n=1$ is $\frac{d(x)}{dx}=1$ which is true. Next assume that $\frac{d(x^n)}{dx}=nx^{n-1}$ is true for some $n \in N$. Then consider the $n+1$ case $\frac{d(x^{n+1})}{dx}=\frac{d(x^n \times x)}{dx}$ and use the product rule

$x\frac{d(x^n)}{dx}+x^n\frac{d(x)}{dx}=x \times n \times x^{n-1}+x^n=(n+1)x^n$ which is true by the induction hypothesis.

Therefore $\frac{d(x^n)}{dx}=nx^{n-1}$ holds by the mathematical induction, for all $n \in N$.

2 **a** $f(2)=2^3+2^2-5\times 2-2=8+4-10-2=0$

b p=3; q=1

c $r=\frac{-3\pm\sqrt{5}}{2}$

d **i** Arithmetic: 1, 1+r, 1+2r, 1+3r

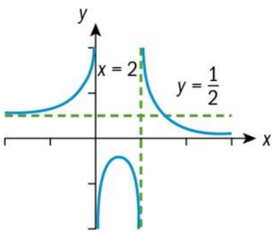
Geometric: 1, r, r^2, r^3

ii r=2, $r=\frac{-3\pm\sqrt{5}}{2}$

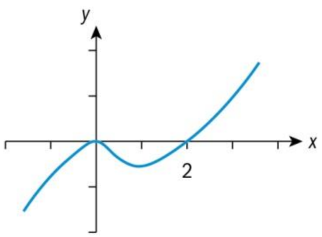
iii $r=\frac{-3\pm\sqrt{5}}{2}$

iv $-265+95\sqrt{5}$

3 **a**



b



4 **a** $f'(x)=24x^2+2rx+s$

$f'(x)=0 \Rightarrow x=\frac{1}{24}\left(-r\pm\sqrt{r^2-24s}\right)$. For two real roots to exist we demand $r^2>24s$.

b $r=12$; $s=-18$, $t=-7$

5 **a** 1C, 2A and 3B A3

b -3 and 0 A1

c -3 A1

6 **a** $-2 \le x \le 3$ A1

b -2 and 0 A1

c $0 \le f(x) \le 2$ A1

d $-2 < x < -1,\ 0 < x < 2$ A2

e any point between 2 and 3 A1

f -1, 0 and 2 A1

7 **a** $f'(x) = 3x^2 + 4x - 4$ M1

$= (3x-2)(x+2)$ A1

b tangent horizontal $\Rightarrow f'(x) = (3x-2)(x+2) = 0$ M1

Normals at these points are vertical lines, so equations are $x = \frac{2}{3}$ and $x = -2$ A1A1

c $f'(2) = 16$ A1

Gradient of normal: $-\frac{1}{16}$ M1A1

$f(2) = 3$ A1

$x + 16y = 50$ (or equivalent) M1A1

d Substitute $x = \frac{2}{3}$ and $x = -2$ into $x + 16y = 50$ M1

$\left(-2, \frac{13}{4}\right)$ and $\left(\frac{2}{3}, \frac{37}{12}\right)$ A1A1

8 **a** $P(x,\ x^2 - 3x)$ A1

b Let $p(x)$ represent the perimeter of triangle *OAP*, as a function of *x.*

$p(x) = |OP| + |PA| + |AO|$ M1

$p(x) = \sqrt{x^2 + (x^2-3x)^2} + \sqrt{(x-4)^2 + (x^2-3x+1)^2} + \sqrt{4^2+1}$ A2

Use GDC to find minimum M1

$x = 2.75$ A1

P is the point $(2.75, -0.6875)$. A1

9 a $x = \dfrac{y+2}{y-1}$ M1

$x(y-1) = y+2$

$xy - y = x + 2$ A1

$g^{-1}(x) = \dfrac{x+2}{x-1} = g(x)$ A1

b $f'(x) = \left(1 - \dfrac{1}{x}\right)' = \dfrac{1}{x^2}$ M1A1

$(f \circ g)'(x) = f'(g(x))g'(x)$ M1

$= \left(\dfrac{x-1}{x+2}\right)^2 \cdot \left(-\dfrac{3}{(x-1)^2}\right) = -\dfrac{3}{(x+2)^2}$ A1

10 a We require $\dfrac{2x}{4-x} \geq 0$ and $x \neq 4$ R1

$\dfrac{2x}{4-x} \geq 0 \Rightarrow 2x \geq 0$ and $4 - x > 0$ A1

(this is because both $2x \leq 0$ and $4 - x < 0$ is impossible) R1

Hence $0 \leq x < 4$. AG

b $f'(x) = \dfrac{1}{2}\left(\dfrac{2x}{4-x}\right)^{-\frac{1}{2}} \cdot \left(\dfrac{2x}{4-x}\right)'$ M1A1

$= \dfrac{1}{2}\left(\dfrac{4-x}{2x}\right)^{\frac{1}{2}} \cdot \dfrac{2(4-x)+2x}{(4-x)^2}$ A1

$= \dfrac{4}{(4-x)^2}\sqrt{\dfrac{4-x}{2x}}\,,\ 0 < x < 4$ A1

c There are no values of x on the interval $0 < x < 4$ for which

$f'(x) = \dfrac{4}{(4-x)^2}\sqrt{\dfrac{4-x}{2x}} = 0$ A1

Hence, the function is either strictly increasing, or strictly decreasing, on the entire domain.
R1

Pick any point in the domain, e.g. $x = 1$.

Then $f'(1) = \frac{4}{(4-1)^2}\sqrt{\frac{4-1}{2\times 1}} = 0.554 > 0$ R1

So the function is increasing on the entire domain. A1

d $f(0) = 0$ A1

$\lim_{x\to 4^-} f(x) = +\infty$ A1

$0 \le x < 4 \Rightarrow f(x) \ge 0$ A1

5 Analysing data and quantifying randomness: statistics and probability

1 Let $f(x) = x_n,\ n = 1, 2, 3...10$. If the standard deviation of $f(x)$ is 10, calculate the standard deviation of

a $3f(x)$

b $\sum_{n=1}^{10} 2f(x) + 1$

2 In a family there are two children, Peter and David. Peter receives 100 dollars on his first birthday and on subsequent birthdays he receives an additional 10% to the previous year, David receives 100 dollars on his first birthday and an additional 15 dollars to the previous each year.

a Draw a table to represent the monies received each year for the first 15 years.

b Calculate the mean and variance of each child receives during the first 15 years.

c When does Peter receive more money than David?

d Who would receive the most money after 5 years, 10, years, 15 years.

3 Show that the two formulae for the product moment correlation coefficient

$r = \dfrac{\Sigma(x-\bar{x})(y-\bar{y})}{\sqrt{\left\{\Sigma(x-\bar{x})^2\,\Sigma(y-\bar{y})^2\right\}}}$ and $r = \dfrac{\Sigma xy - (\Sigma x \Sigma y)/n}{\sqrt{\left\{\left[\Sigma x^2 - \dfrac{(\Sigma x)^2}{n}\right]\left[\Sigma y^2 - \dfrac{(\Sigma y)^2}{n}\right]\right\}}}$ are equivalent.

4 Over a period of 200 weeks a company recorded the number of errors made by their staff. The results are given in the table below.

No of errors, x	Number of weeks, f
0	5
1	22
2	46
3	38
4	31
5	23
6	16
7	11
8	6
9	2

Here $\sum fx = 706$, $\sum fx^2 = 3280$.

a Construct a data representation of this information.

b State the modal value and calculate the mean, median and standard deviation of the errors per week.

c There are two possible tests for skewness

$\frac{3(\text{mean} - \text{median})}{\text{standard deviation}}, \frac{(\text{mean} - \text{mode})}{\text{standard deviation}}$. Calculate and compare these two measures for skewness.

d State how this skewness is reflected in the shape of your graph.

5 A linear function is stated as $f(x) = ax + b$. It transforms a set A, such that $A = \{1, 2, 3, 5, 8, 11\}$ into a set B. If f(5)=13 and f(1) =5,

a Find the values of a and b.

b Calculate the mean and variance of A.

c Calculate the mean and variance of B.

d A new set C is created by adding an element, k to set A. The mean of set C is three times the mean of set A. Find the value k.

e Find the variance of set C.

6 The following functions define frequency distributions. For each one draw the frequency distribution table and calculate the mean and standard deviation for the distributions.

a $f(x) = 3e^{-0.2x}$, $x = 0, 1, 2 \ldots 8$

b $f(x) = 2x^2 + 4x - 3$, $x = 0, 1, 2 \ldots 8$

7 For a set of 20 pairs of observations of the variables x and y, it is given that

$\sum x = 250$ and $\sum y = 140$. The regression line y on x passes through the point (15,10). Find the equation of the regression line and use it to estimate the value of y when x = 10.

8 It is suspected that two quantities A and B are related by the formula $B = sA^t$, where s and t are constants. Observations on A and B are recorded in the table

A	13	16	20	25	32	40	50	60
B	71	40	50	32	24	31	25	16

a Plot a scatter graph of $\log_{10} B$ against $\log_{10} A$.

b Estimate the equation of the regression line of $\log_{10} B$ on $\log_{10} A$.

c Use your results to find estimates for s and t.

Exam-style questions

9 Classify the following data as either discrete or continuous.

a Number of emails received in a day. (1)

b Maximum temperature during the day. (1)

c Distance walked during a day. (1)

d Length of a telephone call rounded to the nearest minute. (1)

e Time spent cleaning teeth during the day. (1)

f The number of people spoken to during the day. (1)

10 Consider the following set of data $7, 8, 19, 20, 21, 22, 23, 24, 25, 35, 36$.

a Find the

i range

ii interquartile range

iii standard deviation. (3)

b Identify any outliers. (2)

c Remove the outliers from the data and, for the remaining values, find the

i range

ii interquartile range

iii standard deviation. (3)

11 A data set has a mean of $\bar{x}$, mode of m , median of Q_2 , lower quartile of Q_1 , upper quartile of Q_3 and variance of s_n^2 . All the values in the original data set have 2 added to them.

a Find each of the following for the new set of data. Express your answers in terms of the parameters $\bar{x}$, m , Q_2 , Q_1 , Q_3 and s_n^2 from the original data set.

i the mean

ii the mode

iii the median

iv the interquartile range

v the variance. (5)

All the values from the **original** data set are now multiplied by −3.

b Find each of the following for the new set of data. Express your answers in terms of the parameters $\bar{x}$, m , Q_2 , Q_1 , Q_3 and s_n^2 from the original data set

i the mean

ii the mode

iii the median

iv the lower quartile

v the upper quartile

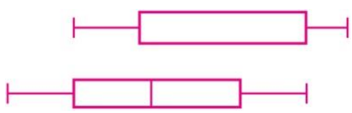
vi the variance. (6)

c **i** All the values from the original data set have 4 added to them and are then multiplied by 5. Find an expression for the new variance.

ii All the values in the original data set are multiplied by 5, and then have 4 added to them. Find an expression for the new variance.

iii Using your answers to parts **i** and **ii**, comment on how variance is affected by both addition and multiplication of each data piece (3)

12 The two box and whisker diagrams below represent the heights of police officers. The top one represents male officers and the bottom one represents female officers. Both are plotted on the same scale axis.



a State what percentage of male officers are taller than all female officers. (2)

b State what percentage of female officers are taller than the shortest male officer. (2)

13 The paired bivariate data below gives the IB HL Maths grade, x, and the HL Physics grade, y, for 10 students.

Maths x	1	3	3	4	5	6	6	7	7	7
Physics y	2	4	3	3	5	6	7	6	7	6

a Calculate the Pearson product-moment correlation coefficient for the above data. (2)

b Calculate the equation of the y on x linear regression line. (2)

c Calculate the equation of the x on y linear regression line. (2)

d Find the angle between the two straight lines found in parts **b** and **c**, giving your answer to the nearest degree. (3)

For the same 10 students the table below gives the IB HL Maths grade, x, and the HL Art grade, z.

Maths x	1	3	3	4	5	6	6	7	7	7
Art z	4	4	5	7	4	7	5	4	5	7

e Calculate the Pearson product-moment correlation coefficient for the above data. (2)

f Calculate the equation of the z on x linear regression line. (2)

g Calculate the equation of Calculate the x on z linear regression line. (2)

h Explain why it would not be particularly valid to use the regression lines found in parts **f** and **g** to make any predictions. (1)

i Find the acute angle between the two straight lines found in parts **f** and **g**, giving your answer to the nearest degree. (3)

j Suggest (with a reason) what happens, in general, to the angle between the y on x linear regression line and the x on y linear regression line as the correlation coefficient r increases from 0 to 1. (2)

k If $r = 1$, state what the angle between these two lines would be and explain why. (1)

Answers

1 **a** 30

b 20

2 **a**

	Peter	David
1	100	100
2	110	115
3	121	130
4	133.1	145
5	146.41	160
6	161.051	175
7	177.1561	190
8	194.8717	205
9	214.3589	220
10	235.7948	235
11	259.3742	250
12	285.3117	265
13	313.8428	280
14	345.2271	295
15	379.7498	310
mean	211.8165	205
variance	7354.076	4200
st dev	85.75591	64.80741

b On his 10^{th} birthday.

c As David receives more money every birthday up to the 10^{th}, he would receive most money after 5 years, and as the difference in the total amounts on the 10^{th} birthday is very small, he would also receive most money after 10 years. After 15 years Peter would receive 3177.248 dollars, and David would receive 3075 dollars. Therefore after 15 years Peter would receive more money than David.

3 $r = \dfrac{\sum(x-\bar{x})(y-\bar{y})}{\sqrt{\left\{\sum(x-\bar{x})^2 \sum(y-\bar{y})^2\right\}}} = \dfrac{\dfrac{\sum(x-\bar{x})(y-\bar{y})}{n}}{\sqrt{\left\{\dfrac{\sum(x-\bar{x})^2}{n}\dfrac{\sum(y-\bar{y})^2}{n}\right\}}}$. Note that

$$\frac{\sum(x-\bar{x})(y-\bar{y})}{n} = \frac{\sum xy}{n} - \frac{\sum \bar{x}y}{n} - \frac{\sum x\bar{y}}{n} + \frac{\sum \bar{x}\bar{y}}{n}$$

$$= \frac{\sum xy}{n} - \bar{x}\bar{y} - \bar{x}\bar{y} + \bar{x}\bar{y}\frac{n}{n}$$

$= \dfrac{\sum xy}{n} - \bar{x}\bar{y}$. Using $\bar{x} = \dfrac{\sum x}{n}$, $\bar{y} = \dfrac{\sum y}{n}$,

$\dfrac{\sum(x-\bar{x})(y-\bar{y})}{n} = \dfrac{\sum xy}{n} - \dfrac{\sum x \sum y}{n^2}$. Similarly

$$\sqrt{\frac{\sum(x-\bar{x})^2}{n}} = \sqrt{\frac{\sum x^2}{n} - \frac{\sum 2\bar{x}x}{n} + \frac{\sum \bar{x}^2}{n}}$$

$$= \sqrt{\frac{\sum x^2}{n} - 2\bar{x}\frac{\sum x}{n} + \bar{x}^2\frac{\sum 1}{n}}$$

$$= \sqrt{\frac{\sum x^2}{n} - 2\frac{\left(\sum x\right)^2}{n^2} + \frac{\left(\sum x\right)^2}{n^2}}$$

$= \sqrt{\dfrac{\sum x^2}{n} - \dfrac{\left(\sum x\right)^2}{n^2}}$. Therefore

$\sqrt{\sum(x-\bar{x})^2} = \sqrt{\sum x^2 - \dfrac{\left(\sum x\right)^2}{n}}$ and $\sqrt{\sum(y-\bar{y})^2} = \sqrt{\sum y^2 - \dfrac{\left(\sum y\right)^2}{n}}$. So

$$r = \frac{\sum xy - \left(\sum x \sum y\right)/n}{\sqrt{\left\{\left[\sum x^2 - \dfrac{\left(\sum x\right)^2}{n}\right]\left[\sum y^2 - \dfrac{\left(\sum y\right)^2}{n}\right]\right\}}}.$$

4 **a** Draw a histogram.

b Mode = 2, mean = 3.53, median = 3, standard deviation = 1.985.

c 0.801, 0.771

d The answer to this question will vary depending on the graph.

5 **a** a=2, b=3

b 5, 12.(3)

c 13, 49.(3)

d 26

e $64\frac{4}{7}$ (64.571)

6

a

0	3
1	2.456
2	2.011
3	1.646
4	1.348
5	1.104
6	0.904
7	0.74
8	0.606
mean	1.535
st dev	0.770809

b

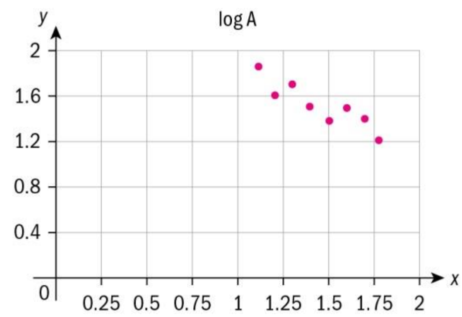
0	-3
1	3
2	13
3	27
4	45
5	67
6	93
7	123
8	157
mean	58.33333
st dev	52.94861

7 $y = 1.2x - 8$, $y = 4$.

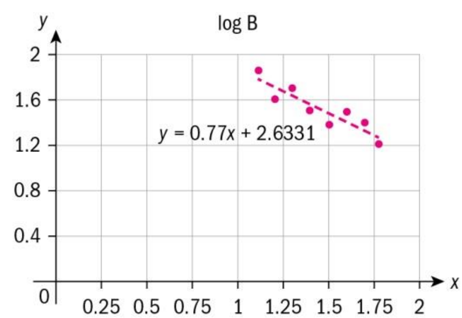
8 a

A	B	log A	log B
13	71	1.113943	1.851258
16	40	1.20412	1.60206
20	50	1.30103	1.69897
25	32	1.39794	1.50515
32	24	1.50515	1.380211
40	31	1.60206	1.491362
50	25	1.69897	1.39794
60	16	1.778151	1.20412

and means are 1.4502 and 1.5164.



b



Gradient = -0.77.

c $B = sA^t$

$\log B = \log(sA^t)$
$\log B = \log s + t\log A$
$y = \log s + tx$

Using the value for the gradient t=-0.77 and the mean values (1.45,1.52),

$1.52 = \log s + (-.77)(1.45)$

$2.6365 = \log_{10} s$
$s = 433$

Therefore, s=433, t=-0.77.

9 **a** Discrete A1

b Continuous A1

c Continuous A1

d Discrete A1

e Continuous A1

f Discrete A1

10 a **i** $36 - 7 = 29$ **ii** $25 - 19 = 6$ **iii** $8.60(3sf)$ A1A1A1

b $1.5 \times 6 = 9$ so $7, 8, 35, 36$ are outliers M1A1

c **i** $25 - 19 = 6$ **ii** $24 - 20 = 4$ **iii** 2 A1A1A1

11 a **i** $\bar{x}+2$ **ii** $m+2$ **iii** Q_2+2

iv Q_3-Q_1 **v** s_n^2. A1A1A1A1A1

b **i** $-3\bar{x}$ **ii** $-3m$ iii $-3Q_2$

iv $-3Q_3$ **v** $-3Q_1$ **vi** $9s_n^2$. A1A1A1A1A1A1

c **i** $25s_n^2$ **ii** $25s_n^2$ A1A1

iii Variance affected only when each data piece is multiplied by a constant, not by adding, so answers to **i** and **ii** are the same. R1

12 a 25% A2

b 75% A2

13 a $r=0.922(3sf)$ A2

b $y=0.794x+1.01(3sf)$ A2

c $x=1.07y-0.339(3sf)$ A2

d 1st line has gradient of 0.794, 2nd line has gradient of $\frac{1}{1.07}=0.935$

$\arctan 0.935-\arctan 0.794 \simeq 5^\circ$ A1M1A1

e $r=0.332(3sf)$ A2

f $z=0.211x+4.17(3sf)$ A2

g $x=0.526z+2.17$ A2

h As the r value is small there is only weak correlation and so the regression lines are not particularly valid. R1

i 1st line has gradient of 0.211 2nd line has gradient of $\frac{1}{0.526}=1.90$

$\arctan 1.90-\arctan 0.211 \simeq 50^\circ$ A1M1A1

j As r increases the linear correlation becomes stronger so the lines become closer to each other R1

and thus the angle between them decreases. R1

k This would be perfect linear correlation, so the lines would be the same and the angle between them would be 0. R1

6 Relationships in space: geometry and trigonometry

1 Solve the simultaneous equations for all values of x and y, where $0 \le x \le 360°$ and $0 \le y \le 360°$

$$\sin(x+y) = \frac{1}{\sqrt{2}}$$

$$\cos 2x = -\frac{1}{2}$$

2 Let $f(x) = \arctan(x-1)$, $x>0$.

a Sketch the function and state the range of x.

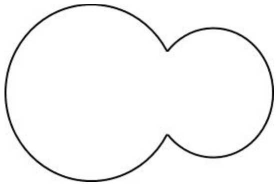
b Given that there exists a normal to the function where the equation of the normal is $y = -2x + c$, calculate the value of c.

3 A student is running along a straight line road in the direction 053°. The student observes a house on a bearing of 037°. 800 meters further along the straight road, the bearing of the house is now 296°.

a Calculate the distance from the student to the house at this point.

b Calculate the shortest distance from the house to the road.

4 Two discs of radii 3 cm and 4 cm are laid on a table such that their centres are 5 cm apart.



Find the perimeter of the outline of the shape.

Exam-style questions

5 A sample of 10 heights of the tide against a sea wall are collected during a 12-hour period. They are as follows:

0.75 m, 1.81 m, 0.93 m, 1.26 m, 1.34 m, 0.62 m, 1.05 m, 0.67 m, 1.21 m, 0.88 m

a Calculate an estimate of the standard deviation for this sample. (2)

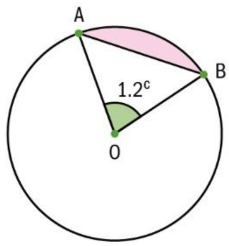
It is suggested that the tide height can be modelled by the equation

$$h(t) = 0.6\sin\left(\frac{\pi t}{8}\right) + 1.3$$

where h is the height in metres and t is the time in hours after midday.

b Calculate the period of time, in hours and minutes, that the tide is above 1.5 metres.(11)

6 The shaded segment of the circle, formed by a central angle of 1.2 radians, has an area of 3.91 cm^2.



Calculate the arc length AB . Give your answer correct to three significant figures. (6)

7 If $\sin A = \frac{1}{3}$, $0 < A < 90°$ and $\sin B = -\frac{2}{5}$, $180° < B < 270°$, and

$z = \sec A + i \cot B$, calculate the exact value of $|z|$. (8)

8 **a** By using a suitable identity involving sine and cosine, show that $1 + \tan^2 x \equiv \sec^2 x$. (1)

b By differentiating from first principles, show that $\frac{d(\tan x)}{dx} = \sec^2 x$. (8)

c Hence, or otherwise, show that $\frac{d(\arctan(\frac{x}{a}))}{dx} = \frac{a}{a^2 + x^2}$ (6)

d State the range of $f(x) = \arctan(\frac{x}{a}), x \in \mathbb{R}$. Justify your answer. (1)

9 The tangent to $y = \frac{x-4}{x}$ when $x = -2$ intersects the x-axis point P.

The tangent to $y = 3\cos 2x$ when $x = \frac{\pi}{12}$ intersects the x-axis at point Q.

The two tangents intersect each other at point R.

Calculate the area of the triangle PQR. Give your answer to 3 significant figures. (17)

Answers

1 If $x = 60^0$ $y = 75^0, 345^0$

If $x = 120^0$ $y = 15^0, 285^0$

If $x = 240^0$ $y = 165^0, 255^0$

If $x = 300^0$ $y = 105^0, 195^0$

2 $c = 4 + \frac{\pi}{4}$

3 **a** 225m

b 200m

4 33.3cm

5 **a** GDC gives $s = 0.364$ (M1)A1

b $0.6\sin\left(\frac{\pi t}{8}\right) + 1.3 = 1.5$ M1

$0.6\sin\left(\frac{\pi t}{8}\right) = 0.2$ M1

$\sin\left(\frac{\pi t}{8}\right) = \frac{1}{3}$ M1

$\frac{\pi t}{8} = 0.3398..., 2.8017...$ A2

$t = 0.865, 7.135$ A2

6.27 hours = 6 hours and 16 minutes (M1) A1

6 Use of $\frac{1}{2}r^2\theta - \frac{1}{2}r^2\sin\theta = \frac{1}{2}r^2(\theta - \sin\theta)$ (M1)

$\frac{1}{2}r^2(1.2 - \sin 1.2) = 3.91$ M1

$r^2 = 29.18$ (M1)

$r = 5.402$ A1

Arc length = $r\theta = 5.402 \times 1.2$ (M1)

Arc length = 6.48 cm A1

7 Use of $\sin^2 x + \cos^2 x = 1$, or right-angled triangle, to find cosA and/or cosB M1

$\cos A = \frac{2\sqrt{2}}{3}, \cos B = -\frac{\sqrt{21}}{5}$ A2

$\sec A = \frac{3\sqrt{2}}{4}$ A1

$\cot B = \frac{\cos B}{\sin B} = \frac{\sqrt{21}}{2}$ (M1)A1

$|z| = \sqrt{\left(\frac{3\sqrt{2}}{4}\right)^2 + \left(\frac{\sqrt{21}}{2}\right)^2}$ M1

$|z| = \frac{\sqrt{102}}{4}$ A1

8 **a** $\sin^2 x + \cos^2 \equiv 1$ M1

$\frac{\sin^2 x}{\cos^2 x} + \frac{\cos^2}{\cos^2 x} \equiv \frac{1}{\cos^2 x}$ A1

$\Rightarrow 1 + \tan^2 x \equiv \sec^2 x$ AG

b $\frac{d(\tan x)}{dx} = \lim_{h\to 0}\frac{\tan(x+h) - \tan x}{h}$ M1

$= \lim_{h\to 0}\frac{\frac{\sin(x+h)}{\cos(x+h)} - \frac{\sin x}{\cos x}}{h}$ M1

$= \lim_{h\to 0}\frac{\sin(x+h)\cos x - \cos(x+h)\sin x}{h\cos x\cos(x+h)}$ M1

$= \lim_{h\to 0}\frac{\sin(x+h-x)}{h\cos x\cos(x+h)}$ M1

$= \lim_{h\to 0}\frac{\sin h}{h\cos x\cos(x+h)}$ M1

$= \lim_{h\to 0}\frac{\sin h}{h} \times \lim_{h\to 0}\frac{1}{\cos x\cos(x+h)}$ M1

$= 1 \times \frac{1}{\cos^2 x}$ M1

$= \sec^2 x$ A1

c Let $y = \arctan\left(\frac{x}{a}\right)$

$\Rightarrow \tan y = \frac{x}{a}$ M1

Differentiating implicitly, $\sec^2 y\frac{dy}{dx} = \frac{1}{a}$ M1

By part a, $(1 + \tan^2 y)\frac{dy}{dx} = \frac{1}{a}$ M1

$\left(1+\frac{x^2}{a^2}\right)\frac{dy}{dx}=\frac{1}{a}$ M1

$\frac{dy}{dx}=\frac{\frac{1}{a}}{\left(1+\frac{x^2}{a^2}\right)}=\frac{a}{a^2+x^2}$ M1 A1

d $-\frac{\pi}{2}<f(x)<\frac{\pi}{2}$. A1

We have to restrict the domain of the function $\tan\left(\frac{x}{a}\right)$ to $-\frac{\pi}{2}<\frac{x}{a}<\frac{\pi}{2}$ for it to be one-to-one, and hence this is the range of the function $f(x)=\arctan(\frac{x}{a}), x\in\mathbb{R}$. R1

9 First find tangent to $y=\frac{x-4}{x}$ at $x=-2$.

$\frac{dy}{dx}=\frac{x-(x-4)}{x^2}=\frac{4}{x^2}$ M1

when $x=-2,\ y=3,\ \frac{dy}{dx}=1$ A1A1

tangent is $y-3=(x+2)$ M1

$y=x+5$ A1

Now find tangent to $y=3\cos 2x$ when $x=\frac{\pi}{12}$

$\frac{dy}{dx}=-6\sin 2x$ M1

when $x=\frac{\pi}{12},\ y=\frac{3\sqrt{3}}{2},\ \frac{dy}{dx}=-3$ A1A1

tangent is $y-\frac{3\sqrt{3}}{2}=-3\left(x-\frac{\pi}{12}\right)$ A1

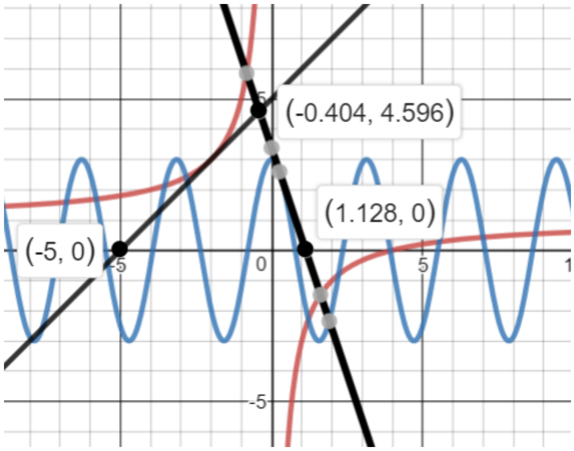
Solve $0=x+5$ to find P $\Rightarrow$ P $(-5, 0)$ M1A1

Solve $-\frac{3\sqrt{3}}{2}=-3\left(x-\frac{\pi}{12}\right)$ to find Q $\Rightarrow Q(1.128, 0)$ M1A1

Substitute $x=y-5$ into $y-\frac{3\sqrt{3}}{2}=-3\left(x-\frac{\pi}{12}\right)$ to find y-value of intersection of tangents

$y-\frac{3\sqrt{3}}{2}=-3\left(y-5-\frac{\pi}{12}\right)$ M1

$y=4.596...$ A1



Area = $\dfrac{6.128 \times 4.596}{2} = 14.08\, units^2$ M1 A1

7 Generalizing relationships: exponents, logarithms and integration

1 **a** Given that $h(x) = g \circ f(x)$, where $f(x) = 3x + \ln 2$ and $g(x) = e^{2x}$ show that $h(x) = 4e^{6x}$.

b Sketch the graph of $h(x)$ and indicate the y-intercept and any asymptotes on your sketch.

c Find the equation of the tangent to $h(x)$ at the point where $x = 0$.

d Find the area bounded by this tangent and the x- and y-axes.

2 **a** Determine $\int x \cos 2x \, dx$.

b Use your result from part **a** to find $\int x \sin^2 x dx$.

3 **a** Prove the identity: $\frac{1}{\log_2 x} + \frac{1}{\log_3 x} + \frac{1}{\log_4 x} + \ldots + \frac{1}{\log_{100} x} = \frac{1}{\log_{100!} x}$

b Given that $e^y = e^x + \frac{1}{e^x}$, prove that $\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^2 = 1$.

Exam-style questions

4 Consider the function $f(x) = \sin x$.

a Let $F(t) = \int_{-\pi}^{t} f(x) dx$.

i Find an expression for $F(t)$ which does not involve an integral.

ii Show that $\frac{d}{dt} F(t) = f(t)$. (4)

b Calculate $\int_{-\pi}^{\pi} f(x) dx$ and $\int_{-\pi}^{\pi} |f(x)| dx$. (3)

c **i** State the range of $f(x)$ and $|f(x)|$.

ii State whether the functions $f(x)$ and $|f(x)|$ are many-to-one or one-to-one for $x \in \mathbb{R}$.

iii With x now restricted to the domain $-\frac{\pi}{2} \le x \le \frac{\pi}{2}$, comment on any change to your classifications of $f(x)$ and $|f(x)|$ in part **ii**. (4)

d Determine the domain and range of $h(x) = f(ax + b) + c$ for $-\pi \le x \le \pi$. Give your answers in terms of a, b and c. (2)

e For $-\pi \le x \le \pi$,

i find the stationary points of $h(x) = f\left(\frac{1}{2}x + \frac{\pi}{4}\right) - 1$

ii sketch the curve

iii find the exact value of the definite integral of $h(x)$ on the interval $-\pi \le x \le \pi$. (5)

5 The hyperbolic functions sinh and cosh are defined as $\sinh x = \dfrac{e^x - e^{-x}}{2}$ and $\cosh x = \dfrac{e^x + e^{-x}}{2}$.

a Show that $\dfrac{d}{dx}\sinh x = \cosh x$ and $\dfrac{d}{dx}\cosh x = \sinh x$. (3)

b Using the substitution $x = \sinh u$, integrate $\int \dfrac{1}{\sqrt{x^2+1}}dx$. (5)

c Find $\int x^2 \sinh x dx$. (6)

The hyperbolic function tanh is defined as $\tanh x = \dfrac{\sinh x}{\cosh x} = \dfrac{e^{2x}-1}{e^{2x}+1}$.

d Write an expression for the inverse function $\tanh^{-1} x$ using logarithms, and differentiate it. Hence, find the exact value of $\int_0^{\frac{1}{2}} \dfrac{1}{1-x^2}dx$. (8)

6 **a** A set of data is made up of the first seven terms of an Arithmetic Progression, with first term 4 and common difference of 3.

For this data, find

i the median

ii the mean

iii the interquartile range. (6)

b A set of data is made up of the first seven terms of a geometric progression, with first term 1 and common ratio 2.

For this data, find

i the median

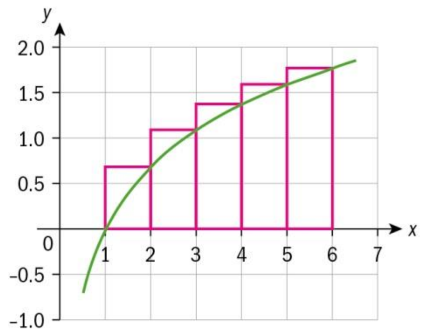
ii the mean (giving your answer as a fraction)

iii the interquartile range. (6)

c State which of these two data sets has mean equal to its median. Suggest a reason for this. (2)

7 **a** Using integration by parts, show that $\int \ln x dx = x\ln x - x + c$ (3)

b The graph below shows how the area under the curve of $y = \ln x$ can be approximated by a sum of rectangles of width 1.



i By comparing the sum of the areas of $(n - 1)$ rectangles with the true area under the graph between $x = 1$ and $x = n$, show that $\ln n! > n\ln n - n + 1 = \ln\left(\left(\frac{n}{e}\right)^n e\right)$. (5)

ii Hence, deduce that $\ln n! > \ln\left(\left(\frac{n}{e}\right)^n e\right)$ (2)

c Prove by induction that $\int_0^\infty x^n e^{-x} dx = n!$ for $n = 0, 1, 2, 3, \ldots$ (you may assume $0! = 1$) (7)

8 A geometric series has first term u_1 and common ratio e^{-x} for a given value of x.

a Write down an expression for the general term of the series, $u_n(x)$, and for the sum of the first n terms, $S_n(x)$. (2)

b Integrate $u_n(x)$ for **i** $n = 1$ and **ii** $n > 1$. (2)

c Hence show that $\int \frac{1 - e^{-nx}}{1 - e^{-x}} dx = x - \sum_{k=1}^{n-1} \frac{1}{ke^{kx}} + c$ (you may interchange the order of integration and summation without any justification). (5)

Answers

1 a $h(x) = g \circ f(x)$

$= g(3x + \ln 2)$

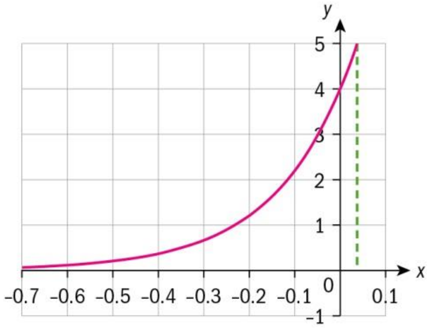
$= e^{2(3x+\ln 2)}$

$= e^{6x} \times e^{2\ln 2}$

$= e^{6x} \times e^{\ln 4}$

$= 4e^{6x}$

b



c $h'(x) = 24e^{6x} \Rightarrow h'(0) = 24$

Equation of tangent: $y = 24x + c$, At $(0,4)$: $y = 24x + 4$

d Tangent meets x-axis at $\left(-\frac{1}{6}, 0\right)$

Area of triangle = $\frac{1}{2} \times \frac{1}{6} \times 4 = \frac{1}{3}$

2 a Integrating by parts

$$\int x\cos 2x\,dx = \frac{1}{2}x\sin 2x - \frac{1}{2}\int \sin 2x\,dx = \frac{1}{2}x\sin 2x + \frac{1}{4}\cos 2x + c$$

b
$$\left.\begin{aligned} &\cos 2x = 1 - 2\sin^2 x \\ &\Rightarrow \sin^2 x = \frac{1-\cos 2x}{2} \end{aligned}\right\} \Rightarrow \int x\sin^2 x\,dx = \frac{1}{2}\int (x - x\cos 2x)\,dx$$

$$= \frac{x^2}{4} - \frac{1}{2}\left(\frac{1}{2}x\sin 2x + \frac{1}{4}\cos 2x\right) + c = \frac{1}{4}\left(x(x - \sin 2x) - \frac{1}{2}\cos 2x\right) + c$$

3 a

$$\frac{1}{\log_2 x} + \frac{1}{\log_3 x} + \frac{1}{\log_4 x} + \ldots + \frac{1}{\log_{100} x}$$
$$= \log_x 2 + \log_x 3 + \log_x 4 + \ldots + \log_x 100$$
$$= \log_x 1 + \log_x 2 + \log_x 3 + \log_x 4 + \ldots + \log_x 100$$
$$= \log_x 1 \times 2 \times 3 \times 4 \times \ldots \times 100$$
$$= \log_x 100! = \frac{1}{\log_{100!} x}$$

b $e^y = e^x + \frac{1}{e^x} = e^x + e^{-x}$

$e^y \frac{dy}{dx} = e^x - e^{-x}$

$\Rightarrow e^y \frac{d^2y}{dx^2} + \frac{dy}{dx} \times e^y \frac{dy}{dx} = e^x + e^{-x}$

$\Rightarrow e^y \left(\frac{d^2y}{dx^2} + \left(\frac{dy}{dx} \right)^2 \right) = e^y$

$\Rightarrow \left(\frac{d^2y}{dx^2} + \left(\frac{dy}{dx} \right)^2 \right) = 1$

4 a i $F(t) = \int_{-\pi}^{t} \sin x dx = [-\cos x]_{-\pi}^{t} = -\cos t + \cos(-\pi) = -1 - \cos t$ M1A1

ii $\frac{d}{dt} F(t) = \frac{d}{dt}(-1 - \cos t) = \sin t = f(t)$ M1A1AG

b $\int_{-\pi}^{\pi} f(x) dx = \int_{-\pi}^{\pi} \sin x dx = [-\cos x]_{-\pi}^{\pi} = -\cos \pi + \cos(-\pi) = 0$ A1

$\sin x \geq 0 \quad 0 \leq x \leq \pi$
$\sin x \leq 0 \quad -\pi \leq x \leq 0$ $\Rightarrow$

$\int_{-\pi}^{\pi} |f(x)| dx = \int_{-\pi}^{\pi} |\sin x| dx = \int_{0}^{\pi} \sin x dx + \int_{-\pi}^{0} -\sin x dx = [-\cos x]_{0}^{\pi} + [\cos x]_{-\pi}^{0} = 4$ M1A1

c i The range of $f(x) = \sin x$ is $-1 \leq f(x) \leq 1$ and the range of $g(x) = |\sin x|$ is $0 \leq g(x) \leq 1$. A1

ii Both $f(x)$ and $|f(x)|$ are many-to-one. A1

iii Restricted to the domain $-\frac{\pi}{2} \leq x \leq \frac{\pi}{2}$, $f(x)$ is now one-to-one and $|f(x)|$ is many-to-one. A1A1

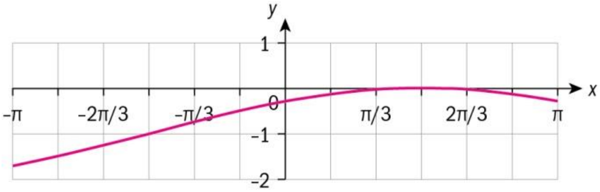
d Domain is $-a\pi + b \leq x \leq a\pi + b$

Range is $-1 + c \leq h(x) \leq 1 + c$. A2

e i $h'(x) = f'\left(\frac{1}{2}x + \frac{\pi}{4}\right) \frac{d}{dx}\left(\frac{1}{2}x + \frac{\pi}{4}\right) = \frac{1}{2}\cos\left(\frac{1}{2}x + \frac{\pi}{4}\right) = 0 \Rightarrow \frac{1}{2}x + \frac{\pi}{4} = \frac{\pi}{2} \Rightarrow x = \frac{\pi}{2}$

M1A1

ii



(A1)

iii $\int_{-\pi}^{\pi}\left(\cos\left(\frac{1}{2}x+\frac{\pi}{4}\right)-1\right)dx=\left[2\sin\left(\frac{1}{2}x+\frac{\pi}{4}\right)-x\right]_{-\pi}^{\pi}$

$=2\sin\left(\frac{\pi}{2}+\frac{\pi}{4}\right)-\pi-2\sin\left(\frac{-\pi}{2}+\frac{\pi}{4}\right)+(-\pi)=2\left(\sqrt{2}-\pi\right)$ M1A1

5 a $\frac{\mathrm{d}}{\mathrm{d}x}\sinh x=\frac{\mathrm{d}}{\mathrm{d}x}\frac{e^x-e^{-x}}{2}=\frac{e^x-(-1)e^{-x}}{2}=\frac{e^x+e^{-x}}{2}=\cosh x$ M1A1AG

$\frac{\mathrm{d}}{\mathrm{d}x}\cosh x=\frac{\mathrm{d}}{\mathrm{d}x}\frac{e^x+e^{-x}}{2}=\frac{e^x+(-1)e^{-x}}{2}=\frac{e^x-e^{-x}}{2}=\sinh x$ A1AG

b $x=\sinh u$, $dx=\cosh u\ du$, $u=\sinh^{-1}x$ M1

$x^2+1=\sinh^2 u+1=\left(\frac{e^u-e^{-u}}{2}\right)^2+1=\frac{e^{2u}-2+e^{-2u}}{4}+1=\frac{e^{2u}+2+e^{-2u}}{4}=\left(\frac{e^u+e^{-u}}{2}\right)^2=\cosh^2 u$

M1A1

$\int\frac{1}{\sqrt{x^2+1}}dx=\int\frac{1}{\sqrt{\cosh^2 u}}\cosh u du=\int du=u+c=\sinh^{-1}x+c$. M1A1

c Use integration by parts with $u=x^2$, $\frac{\mathrm{d}u}{\mathrm{d}x}=2x$ and $\frac{\mathrm{d}v}{\mathrm{d}x}=\sinh x$, $v=\cosh x$ M1A1

$\int x^2\sinh x dx=x^2\cosh x-\int 2x\cosh x dx$ A1

Now use integration by parts again to find $\int 2x\cosh x dx$ M1

Here $u=2x$, $\frac{\mathrm{d}u}{\mathrm{d}x}=2$ and $\frac{\mathrm{d}v}{\mathrm{d}x}=\cosh x$, $v=\sinh x$ A1

$\int x^2\sinh x dx=x^2\cosh x-[2x\sinh x+\int 2\sinh x dx]=x^2\cosh x-2x\sinh x+2\cosh x+c$ A1

d $y=\tanh^{-1}x$

$x=\tanh y=\frac{e^{2y}-1}{e^{2y}+1}\Rightarrow x\left(e^{2y}+1\right)=e^{2y}-1$ M1

$\Rightarrow(1-x)e^{2y}=1+x$ A1

$\Rightarrow e^{2y}=\frac{1+x}{1-x}\Rightarrow y=\frac{1}{2}\ln\left(\frac{1+x}{1-x}\right)$ A1

In order to differentiate, easiest to write in the form $y=\frac{1}{2}\ln(1+x)-\frac{1}{2}\ln(1-x)$ M1

$\Rightarrow\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{1}{2(1+x)}-\frac{-1}{2(1-x)}=\frac{1}{(1+x)(1-x)}=\frac{1}{1-x^2}$ A1A1

$\int_0^{\frac{1}{2}}\frac{1}{1-x^2}dx=\left[\frac{1}{2}\ln\left(\frac{1+x}{1-x}\right)\right]_0^{\frac{1}{2}}=\frac{1}{2}\ln\left(\frac{\frac{3}{2}}{\frac{1}{2}}\right)-\frac{1}{2}\ln\left(\frac{1}{1}\right)=\frac{1}{2}\ln 3$ M1A1

6 a i The median is the middle term which is $u_4 = 4 + (4-1)3 = 13$ (M1)A1

ii $\bar{x} = \dfrac{\sum_{i=1}^{7} x_i}{n} = \dfrac{\frac{7}{2}(2 \times 4 + (7-1)3)}{7} = 13$ M1A1

iii $IQR = Q_3 - Q_1 = u_6 - u_2 = 4d = 12$ M1A1

b i The median is the middle term which is $u_4 = 1 \times 2^{4-1} = 8$ (M1)A1

ii $\bar{x} = \dfrac{\sum_{i=1}^{7} x_i}{n} = \dfrac{\left[\dfrac{1 \times (2^7 - 1)}{(2-1)}\right]}{7} = \dfrac{127}{7}$ M1A1

iii $IQR = Q_3 - Q_1 = u_6 - u_2 = 2^5 - 2 = 30$ M1A1

c The arithmetic progression A1

For this data set, a graph of n against u_n would yield a straight line (or equivalent) R1

7 a $u = \ln x$, $\dfrac{du}{dx} = \dfrac{1}{x}$ and $\dfrac{dv}{dx} = 1$, $v = x$ A1

$\int \ln x dx = x \ln x - \int x \dfrac{1}{x} dx = x \ln x - x + c$ M1A1

b i True area under graph is given by $\int_1^n \ln x dx$ M1

$= [x \ln x - x]_1^n = n \ln n - n + 1$, by result from part **a.** A1

The area of the rectangles is $1 \times \ln 2 + 1 \times \ln 3 + \ldots + 1 \times \ln n = \ln n!$. M1A1

Because the curve lies completely within the rectangles, $\ln n! > n \ln n - n + 1$ R1

ii $n \ln n - n + 1 = \ln n^n - \ln e^n + \ln e = \ln\left(\left(\dfrac{n}{e}\right)^n e\right)$ A2

so by part **i**, $\ln n! > \ln\left(\left(\dfrac{n}{e}\right)^n e\right)$ AG

c Let $I_n = \int_0^\infty x^n e^{-x} dx$

$I_0 = \int_0^\infty x^0 e^{-x} dx = \int_0^\infty e^{-x} dx = \left[-e^{-x}\right]_0^\infty = 0 - (-1) = 1 = 0!$ So holds for $n = 0$. A1

Assume true for $n = k$, i.e. that $I_k = \int_0^\infty x^k e^{-x} dx = k!$ M1

Now $I_{k+1} = \int_0^\infty x^{k+1} e^{-x} dx$

Use integration by parts with $u = x^{k+1}$, $\dfrac{du}{dx} = (k+1)x^k$ and $\dfrac{dv}{dx} = e^{-x}$, $v = -e^{-x}$ M1A1

Then $I_{k+1} = \int_0^\infty x^{k+1}e^{-x}dx = \left[-x^{k+1}e^{-x}\right]_0^\infty + (k+1)\int_0^\infty x^k e^{-x}dx = (0-0)+(k+1)I_k = (k+1)!$ A2

The result is true for $n=0$ and, when assumed true for $n=k$, it can be proved true for $n=k+1$. Hence, by induction, the result is true for all n. R1

8 **a** $u_n(x) = u_1 r^{n-1} = u_1 e^{-x(n-1)}$ A1

$S_n(x) = u_1 \frac{1-r^n}{1-r} = u_1 \frac{1-e^{-nx}}{1-e^{-x}}$ A1

b $\int u_1(x)dx = \int u_1 dx = u_1 x + c$ A1

$\int u_n(x)dx = \int u_1 e^{-x(n-1)}dx = -u_1 \frac{1}{n-1}e^{-x(n-1)} + c$ A1

c $\int S_n(x)dx = u_1 \int \frac{1-e^{-nx}}{1-e^{-x}}dx = \int \sum_{k=1}^{n} u_k(x)dx = u_1 \int 1 + \sum_{k=2}^{n} e^{-x(k-1)}dx$ M1A1

$u_1\left(\int 1dx + \sum_{k=2}^{n}\int e^{-x(k-1)}dx\right) = u_1\left(x - \sum_{k=1}^{n-1}\frac{1}{ke^{kx}} + c\right)$ A1A1

So $\int \frac{1-e^{-nx}}{1-e^{-x}}dx = x - \sum_{k=1}^{n-1}\frac{1}{ke^{kx}} + c$ AG

8 Modelling change: more calculus

1 Prove using mathematical induction that $\int x^n dx = \frac{x^{n+1}}{n+1} + c$.

2 It is given that $f(x) = \frac{4}{x+2}, x \neq 2$ and $g(x) = x - 1$. If $h = g \circ f$, find

a h **b** h^{-1} **c** $\int h^{-1}(x)dx$

3 Given that $f(x) = x\cos 3x$,

a find $\int f(x)dx$.

b use your answer to a to find the area enclosed by $f(x)$ and the x-axis in each of the following intervals, giving your answer in terms of π.

i $\frac{\pi}{6} \leq x \leq \frac{3\pi}{6}$ **ii** $\frac{3\pi}{6} \leq x \leq \frac{5\pi}{6}$ **iii** $\frac{5\pi}{6} \leq x \leq \frac{7\pi}{6}$

c Given that the areas in **b** are the first three terms of an arithmetic sequence, find an expression for the total area enclosed by f and the x-axis for the first n terms of the sequence, in terms of n and π.

4 Use mathematical induction to prove that the series $\sum_{r=1}^{n} \frac{1}{(2r-1)(2r+1)}$ is equal to the discrete function $f(n) = \frac{n}{2n+1} n \in \mathbb{Z}^+$.

Exam-style questions

5 Consider the implicit relation $3x^2 + 4y^2 = 7$.

a By making y the subject of the implicit function, find an explicit function for y, where $y>0$. (2)

b Find the largest real domain of the explicit function from part **a**. (3)

c Find an expression for $\frac{dy}{dx}$. (3)

d Find the gradient of the tangent to the graph of the explicit function from part **a** at $x=1$. (2)

6 The sum of the first n terms of a geometric series is given by the formula $S_n = 1 + (1+r) + (1+r)^2 + \ldots + (1+r)^{n-1}$.

a Find an equivalent formula for S_n. (2)

b On January 1 2017, Sadie borrowed \$$P$ from a bank at a fixed annual compound rate of interest r. She makes a repayment of \$$B$ to the bank on January 1 2018, and then another repayment of \$$B$ on January 1 2019. Find, in terms of B, P, and r, the amount Sadie still owes after making the two payments. (2)

c If she continues to pay back \$$B$ on the 1st of January each subsequent year, find and simplify an expression for the amount she owes, in dollars, immediately after the nth payment. (4)

d Assuming that the loan is completely paid back after n years, show that

$$B = \frac{Pr}{1-(1+r)^{-n}} \, .$$ (4)

e Use the formula in **d** to express n in terms of B, P and r. (2)

f Hence, find the number of years (correct to the nearest year) that it takes to repay \$350 000 at 7.2% annual compound interest if each annual repayment is

i **\$28 000** **ii** **\$25 500** (2)

g Explain what happens if annual repayments are \$25 200 or less, and interpret your answer. (3)

7 Consider the differential equation $2x^2\frac{dy}{dx} = x^2 + y^2$, $x>0$, and the initial condition $y=2$ when $x=1$.

a Use the substitution $v = \frac{y}{x}$ to solve the differential equation. Give your answer in the form $y = f(x)$, where $f(x)$ is a function to be determined. (8)

b Using the function $y=f(x)$ from part (a),

i find the equation of the vertical asymptote

ii find the values of the x-intercepts

iii write down an expression for the area between the graph of $y = f(x)$, the x-axis and the lines $x = 1$ and $x = 4$, and find the area of this region. (8)

8 A particle moves in a straight line. Its velocity v, in ms^{-1}, t seconds after passing a fixed point O on the line is $v(t) = t\sin\left(\frac{\pi}{3}t\right)$.

a Find the values of t in the interval $0 \le t \le 6$ when the particle is at rest. (4)

b Find the total distance traveled by the particle in the first six seconds. (9)

9 **a** On the same set of axes, sketch the graphs of $f(x) = 3 + 2x - x^2$ and $g(x) = -2 + \ln(3 + x)$ for $-3 \le x \le 5$. On your sketch, you should clearly label any axes intercepts and local maxima / minima with the *exact* values of their coordinates. You should also label any asymptotes with their equations. (5)

b Shade the region on your graph where $f(x) \ge g(x)$, $x \ge 0$. (2)

Answers

1 The base case for $n = 0$ is $\int dx = x + c$ which is true. Assume $\int x^n dx = \frac{x^{n+1}}{n+1} + c$ is true for some $n \in \mathbb{N}_0$. Then consider

$$\int x^{n+1} dx = x^{n+1} \times x - \int x(n+1)x^n dx$$

$$\int x^{n+1} dx = x^{n+1} \times x + c - (n+1)\int x^{n+1} dx \Rightarrow \int x^{n+1} dx = \frac{x^{n+2}}{n+2} + c\ .$$

We conclude that $\int x^n dx = \frac{x^{n+1}}{n+1} + c$ is true for all $n \in \mathbb{N}_0$.

2 **a** $h(x) = \frac{2-x}{2+x}$

b $h^{-1}(x) = \frac{4}{1+x} - 2, x \neq -1$

c $\int h^{-1}(x)\,dx = 4\ln(1+x) - 2x + c$

3 **a** $\int f(x)\,dx = \frac{1}{3}x\sin 3x + \frac{1}{9}\cos 3x + c$

b **i** $\frac{2\pi}{9}$

ii $\frac{4\pi}{9}$

iii $\frac{6\pi}{9}$

c $\frac{n\pi}{9}(n+1)$

4 The base case $n = 1$ is $\sum_{r=1}^{1} \frac{1}{(2r-1)(2r+1)} = \frac{1}{(2-1)(2+1)} = \frac{1}{3} = \frac{n}{2n+1}$. We assume that $\sum_{r=1}^{n} \frac{1}{(2r-1)(2r+1)} = \frac{n}{2n+1}$ for some $n \in \mathbb{Z}^+$. Consider the $n+1$ case

$$\sum_{r=1}^{n+1} \frac{1}{(2r-1)(2r+1)} = \sum_{r=1}^{n} \frac{1}{(2r-1)(2r+1)} + \frac{1}{(2(n+1)-1)(2(n+1)+1)}$$

$$= \frac{n}{2n+1} + \frac{1}{(2(n+1)-1)(2(n+1)+1)} = \frac{n}{2n+1} + \frac{1}{(2n+1)(2n+3)}$$

$= \frac{n}{2n+1} + \frac{1}{(2n+1)(2n+3)} = \frac{1+n}{3+2n}$. Therefore the result is true for all $n \in \mathbb{Z}^+$ by mathematical induction.

5 **a** $3x^2 + 4y^2 = 7 \Rightarrow y = \dfrac{\sqrt{7-3x^2}}{2}$ (M1)A1

b $7 - 3x^2 \geq 0 \Rightarrow |x| \leq \sqrt{\dfrac{7}{3}}$

$\Rightarrow -\sqrt{\dfrac{7}{3}} \leq x \leq \sqrt{\dfrac{7}{3}}$ M1A2

c $\dfrac{dy}{dx} = \dfrac{1}{2}\left[\dfrac{1}{2}(7-3x^2)^{-\frac{1}{2}}(-6x)\right] = \dfrac{-3x}{2\sqrt{7-3x^2}}$ M1A1A1

d $\left.\dfrac{dy}{dx}\right|_{x=1} = \dfrac{-3(1)}{2\sqrt{7-3(1)^2}} = -\dfrac{3}{4}$ (M1)A1

6 **a** $S_n = \dfrac{(1+r)^n - 1}{(1+r)-1} = \dfrac{(1+r)^n - 1}{r}$ (M1)A1

b Amount she owes each year:

2017	2018	2019
P	$P(1+r)$ -B	$\left(P(1+r)-B\right)(1+r)-B =$ $P(1+r)^2 - B(1+r) - B$

So Sadie owes $P(1+r)^2 - B(1+r) - B$ after making the two repayments. M1A1

c After n payments:

$P(1+r)^n - B(1+r)^{n-1} - B(1+r)^{n-2} - \ldots - B(1+r) - B$ M1

$= P(1+r)^n - B\left((1+r)^{n-1} + (1+r)^{n-2} + \ldots + (1+r) + 1\right)$ A1

$= P(1+r)^n - BS_n$ M1

$= P(1+r)^n - B\left(\dfrac{(1+r)^n - 1}{r}\right)$ (using the result from **a**) A1

d When the loan is completely repaid,

$P(1+r)^n - B\left(\dfrac{(1+r)^n - 1}{r}\right) = 0$ M1

$\Rightarrow P(1+r)^n = B\left(\dfrac{(1+r)^n - 1}{r}\right)$

$\Rightarrow \dfrac{Pr(1+r)^n}{(1+r)^n - 1} = B$ A1

Dividing top and bottom of the LHS by $(1+r)^n$, R1

$$B = \frac{Pr}{1-(1+r)^{-n}}.$$ A1

e $$B = \frac{Pr}{1-(1+r)^{-n}}$$

$$\Rightarrow B - B(1+r)^{-n} = Pr$$ M1

$$\Rightarrow (1+r)^{-n} = \frac{B-Pr}{B}$$

$$\Rightarrow (1+r)^{n} = \frac{B}{B-Pr}$$

$$\Rightarrow n = \log_{(1+r)} \frac{B}{B-Pr}$$ (or similar expression with logs to a different base) A1

f **i** $$n = \log_{1.072} \frac{28\,000}{28\,000 - 350\,000 \cdot 0.072} = 33.1$$

It would take about 33 years to repay the loan. A1

ii $$n = \log_{1.072} \frac{25\,500}{25\,500 - 350\,000 \cdot 0.072} = 64.0$$

It would take about 64 years to repay the loan. A1

g $$n = \log_{1.072} \frac{25200}{25200 - 350000 \cdot 0.072} = \text{undefined}.$$ A1

Since 350 000 × 0.072 = 25 200, the formula would reduce to $n = \log_{1.072}\left(\frac{B}{0}\right)$, which is undefined. R1

If $B \le 25\,200$ only the interest would ever be paid back and the principal would remain the same, i.e., the debt would never be repaid. R1

7 **a** $$2x^2\left(v + x\frac{dv}{dx}\right) = x^2 + v^2x^2$$ M1

$$v + x\frac{dv}{dx} = \frac{1+v^2}{2}$$ (A1)

$$\Rightarrow 2x\frac{dv}{dx} = (v-1)^2$$ A1

$$2\int \frac{dv}{(v-1)^2} = \int \frac{dx}{x}$$ M1

$$\frac{-2}{v-1} = \ln x + c$$ A1

$$\frac{-2}{\ln x + c} = \frac{y}{x} - 1$$

$\Rightarrow \dfrac{-2x}{\ln x + c} = y - x$

$\Rightarrow y = x - \dfrac{2x}{\ln x + c}$ A1

$y = 2, x = 1 \Rightarrow c = -2$ M1

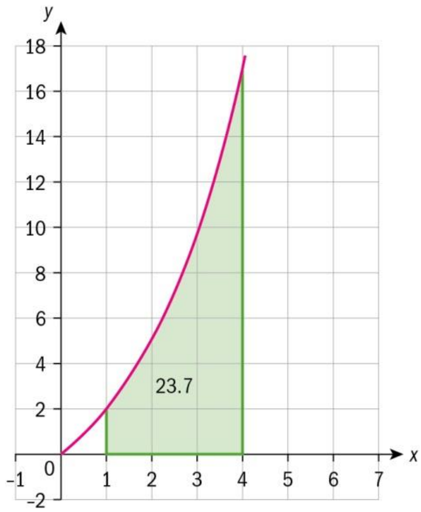
$\Rightarrow y = x - \dfrac{2x}{\ln x - 2}$ A1

b **i** $\ln x - 2 = 0 \Rightarrow x = e^2$ (M1)A1

ii $x - \dfrac{2x}{\ln x - 2} = 0 \Rightarrow \dfrac{x(\ln x - 2) - 2x}{\ln x - 2} = 0$ (M1)

$\Rightarrow x(\ln x - 2 - 2) = 0$
$\Rightarrow x = 0,\ x = e^4$ A1A1

iii



(M1)

$A = \int_1^4 \left(x - \dfrac{2x}{\ln x - 2} \right) dx \approx 23.7$ A1A1

8 **a** $t \sin\left(\dfrac{\pi}{3}t\right) = 0 \Rightarrow t = 0; \sin\left(\dfrac{\pi}{3}t\right) = 0$ (M1)

$\sin\left(\dfrac{\pi}{3}t\right) = 0 \Rightarrow t = 0; t = 3; t = 6$ A3

$\therefore t = 0s, 3s, 6s$

b $d = \int_0^3 t \sin\left(\dfrac{\pi}{3}t\right) dt + \int_3^6 \left| t \sin\left(\dfrac{\pi}{3}t\right) \right| dt$ A1

Using integration by parts on $\int t\sin\left(\frac{\pi}{3}t\right)dt$, M1

$$u = t \Rightarrow du = dt; dv = \int \sin\left(\frac{\pi}{3}t\right)dt \Rightarrow v = -\frac{3}{\pi}\cos\left(\frac{\pi}{3}t\right)$$

$$\int t\sin\left(\frac{\pi}{3}t\right)dt = -\frac{3}{\pi}t\cos\left(\frac{\pi}{3}t\right) + \frac{3}{\pi}\int \cos\left(\frac{\pi}{3}t\right)dt$$ A2

$$= -\frac{3}{\pi}t\cos\left(\frac{\pi}{3}t\right) + \frac{9}{\pi^2}\sin\left(\frac{\pi}{3}t\right)$$ A2

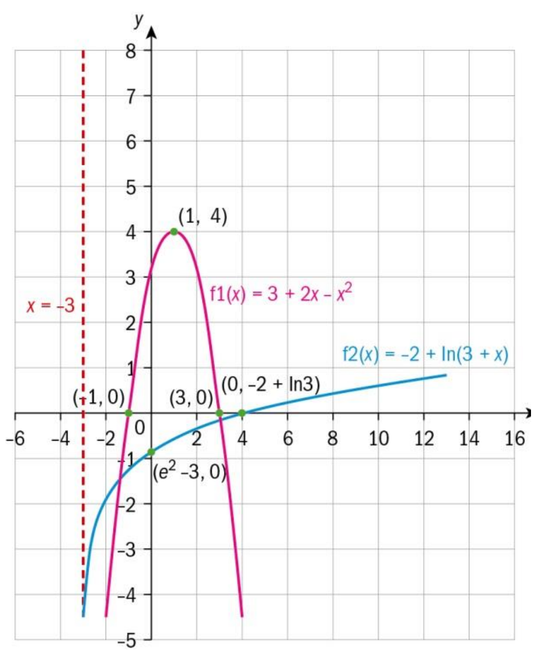
Hence

$$\int_0^3 t\sin\left(\frac{\pi}{3}t\right)dt = \left[-\frac{3}{\pi}t\cos\left(\frac{\pi}{3}t\right) + \frac{9}{\pi^2}\sin\left(\frac{\pi}{3}t\right)\right]_0^3 = \frac{9}{\pi}$$ A1

and $\int_3^6 t\sin\left(\frac{\pi}{3}t\right)dt = \left[-\frac{3}{\pi}t\cos\left(\frac{\pi}{3}t\right) + \frac{9}{\pi^2}\sin\left(\frac{\pi}{3}t\right)\right]_3^6 = -\frac{27}{\pi}$ A1

$$\therefore d = \int_0^3 t\sin\left(\frac{\pi}{3}t\right)dt + \int_3^6 \left|t\sin\left(\frac{\pi}{3}t\right)\right|dt = \frac{9}{\pi} + \left|-\frac{27}{\pi}\right| = \frac{36}{\pi}$$ A1

9 a



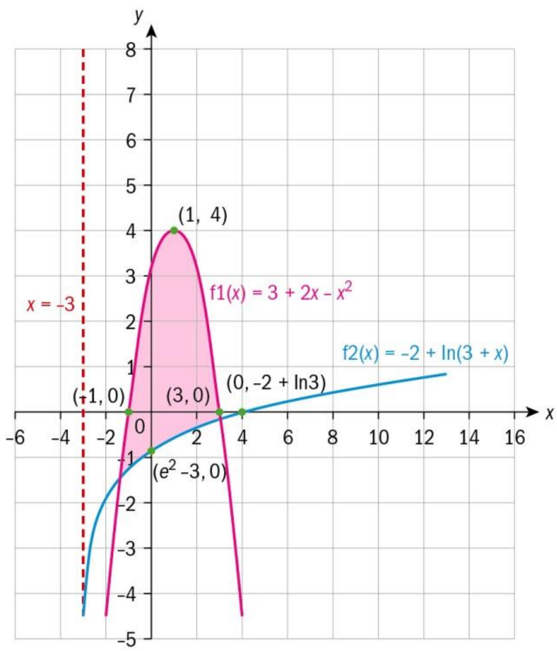
A1A1 for the shape of the graphs

A1 for correctly labelling (-1,0), (1,4), (3,0) on the graph of *f*

A1 for correctly labelling $(0, -2+\ln 3)$ and $\left(e^2-3, 0\right)$ on the graph of *g*

A1 for the asymptote $x = -3$

b



A2

9 Modelling 3D space: vectors

1 The polynomial $f(x) = x^3 - 3x + a$ is divisible by $x + 2$. Find the value of a.

2 Evaluate $\int_{\frac{\pi}{6}}^{\frac{\pi}{4}} \cos(2x)\,dx$.

3 Find the tangent to the curve $x^2 - 4xy + y^2 = 6$ at the point (1,5).

4 The equation $\sin 3\theta + \sin 2\theta = 0$ is given.

a Use the formula $\sin 3\theta = 3\sin\theta - 4\sin^3\theta$ to find all the values of $\cos\theta$ that satisfy the equation.

b Show that $\theta = \frac{2\pi}{5}$ is a solution of the equation.

c Hence find the exact value of $\cos\left(\frac{2\pi}{5}\right)$.

d Show that $\sin\left(\frac{2\pi}{5}\right) = \frac{\sqrt{10 + 2\sqrt{5}}}{4}$.

e Use double angle formulas to find exact values of

i $\cos\left(\frac{4\pi}{5}\right)$ **ii** $\sin\left(\frac{4\pi}{5}\right)$.

In a complex plane a regular pentagon with the vertices $z_1 z_2 z_3 z_4 z_5$ is inscribed in the unit circle such that $z_1 = 1$.

f Find the Cartesian form of complex numbers that represents the next two vertices in the anticlockwise direction.

g Show that $|z_1 - z_2| \cdot |z_1 - z_3| \cdot |z_1 - z_4| \cdot |z_1 - z_5| = 5$.

5 The complex number $z = 4$ in the complex plane is multiplied by the solutions ω_1 and ω_2 of the equation $\omega^2 + \omega + 1 = 0$.

a Find $z \cdot \omega_1$ and $z \cdot \omega_2$.

b Show that z and the results in part a are complex numbers that form an equilateral triangle in the complex plane and find its area.

Given the numbers $z_1, z_2, z_3, \omega \in \mathbb{C}$ such that $\omega^2 + \omega + 1 = 0$.

c If $z_1\omega^2 + z_2\omega + z_3 = 0$ then show that z_1, z_2 and z_3 are vertices of an equilateral triangle in the complex plane.

6 The following set of data is written in the ascending order, $\{2,2,3,a,5,5,6,6,b,7,9\}$. Find the values of a and b given that the mean is 5 and the variance is $\frac{46}{11}$

Exam-style questions

7 **a** Find the Maclaurin series for $\cos\left(x + \frac{\pi}{60}\right)$ up to and including the term in x^2. (4)

b Calculate the value of $\int_0^{0.5} \cos\left(x + \frac{\pi}{60}\right)dx$, giving your answer to 4 decimal places. (2)

c By integrating the Maclaurin series for $\cos\left(x + \frac{\pi}{60}\right)$ found in part **a**, calculate the percentage error of this approximation of the integral. (5)

8 Consider the complex numbers $z_1 = 3 + 5\text{i}$ and $z_2 = -1 + 4\text{i}\sqrt{5}$.

Find simplified expressions for each of the following.

a $|z_1 z_2|$ (3)

b $\left|\frac{z_2}{z_1}\right|$ (3)

c z_1^2 (2)

d $\sqrt{z_2}$ (8)

9 The points $A(-1,2,7)$, $B(3,-3,1)$, $C(2,5,-4)$ and $D(2,7,3)$ form a tetrahedron.

Calculate the exact volume of the tetrahedron. (7)

10 The line $x - 4 = \frac{y+2}{6} = \frac{z-5}{3}$ and the point $A(3,2,5)$ are contained in a plane.

a Find the cartesian equation of the plane. (11)

The plane $33x - 28y + 9z = 88$ also contains point A.

b Find the vector equation of the line of intersection of the two planes. (4)

c Calculate the acute angle between the two planes, giving your answer in radians to 3 significant figures. (5)

11 The line $L: \frac{x+1}{3} = 4 - y = \frac{z-2}{5}$ and the plane $\Pi: 3x + y - 2z = 7$ intersect at point P.

a Find point P. (6)

b Point R has position vector $-3\boldsymbol{i} - 7\boldsymbol{j} + 8\boldsymbol{k}$. Deduce whether point R is closer to the line L or the plane Π. (13)

Answers

1 $a = 2$

2 $\frac{1}{2} - \frac{\sqrt{3}}{4}$

3 $y = 3x + 2$

4 **a** $-1, 1, \frac{1}{4}\left(-1 \pm \sqrt{5}\right)$

b $\sin(3\theta) = \sin\left(3 \times \frac{2\pi}{5}\right) = \sin\left(\frac{6\pi}{5}\right)$, $\sin(2\theta) = \sin\left(\frac{4\pi}{5}\right)$.

$\sin\left(\frac{6\pi}{5}\right) = -\sin\left(2\pi - \frac{6\pi}{5}\right) = -\sin\left(\frac{4\pi}{5}\right) = -\sin(2\theta) = \sin(3\theta)$

c $\frac{\sqrt{5}-1}{4}$

d Using $\sin x^2 + \cos x^2 = 1$ we get $\sin\left(\frac{2\pi}{5}\right) = \sqrt{1 - \cos\left(\frac{2\pi}{5}\right)^2} = \sqrt{1 - \left(\frac{\sqrt{5}-1}{4}\right)^2} = \frac{\sqrt{10+2\sqrt{5}}}{4}$

e **i** $-\frac{\sqrt{5}+1}{4}$ **ii** $\frac{\sqrt{10-2\sqrt{5}}}{4}$

f $z_2 = \frac{\sqrt{5}-1}{4} + \frac{\sqrt{10+2\sqrt{5}}}{4}\mathrm{i}$, $z_3 = -\frac{\sqrt{5}+1}{4} + \frac{\sqrt{10-2\sqrt{5}}}{4}\mathrm{i}$

g The roots are 1, $-\frac{1}{4}\left(1+\sqrt{5}\right) - \frac{\sqrt{10-2\sqrt{5}}}{4}i$, $\frac{1}{4}\left(-1+\sqrt{5}\right) + \frac{\sqrt{10+2\sqrt{5}}}{4}i$,

$\frac{1}{4}\left(-1+\sqrt{5}\right) - \frac{\sqrt{10+2\sqrt{5}}}{4}i$, $\frac{1}{4}\left(-1-\sqrt{5}\right) + \frac{\sqrt{10-2\sqrt{5}}}{4}i$

Find that $|z_1 - z_2| = |z_1 - z_5|$, $|z_1 - z_3| = |z_1 - z_4|$ and

$|z_1 - z_2|^2 = \frac{1}{2}\left(5+\sqrt{5}\right)$, $|z_1 - z_3|^2 = \frac{1}{2}\left(5-\sqrt{5}\right)$. Therefore

$|z_1 - z_2||z_1 - z_3||z_1 - z_4||z_1 - z_5| = \frac{1}{2}\left(5+\sqrt{5}\right)\frac{1}{2}\left(5-\sqrt{5}\right) = \frac{1}{4}(25-5) = 5$

5 **a** $-2+2\sqrt{3}\mathrm{i}, -2-2\sqrt{3}\mathrm{i}$

b $\left|-2+2\sqrt{3}\mathrm{i}-4\right| = \left|-2-2\sqrt{3}\mathrm{i}-4\right| = \left|\left(-2+2\sqrt{3}\mathrm{i}\right)-\left(-2-2\sqrt{3}\mathrm{i}\right)\right| = 4\sqrt{3}$, $Area = 12\sqrt{3}$

c $z_1 = \frac{1}{\omega^2}\left(-\omega z_2 - z_3\right) \Rightarrow |z_1 - z_2| = \left|\frac{1}{\omega^2}\left(-\omega z_2 - z_3\right) - z_2\right|$ and

$\omega^2 = -\omega - 1 \Rightarrow |z_1 - z_2| = \frac{|z_2 - z_3|}{|\omega^2|} = |z_2 - z_3|$. Similarly

$$|z_1 - z_3| = \left|\frac{1}{\omega^2}\left(-\omega z_2 - z_3 - \omega^2 z_3\right)\right| = \left|\frac{z_3 - z_2}{\omega}\right| = |z_3 - z_2|$$

6 $a = 4$, $b = 6$.

7 **a** Stating $\cos x = 1 - \frac{x^2}{2!} \ldots$ M1

Substituting $\left(x + \frac{\pi}{60}\right)$ into expansion above,

$$\cos\left(x + \frac{\pi}{60}\right) = 1 - \frac{\left(x + \frac{\pi}{60}\right)^2}{2!}$$ M1

$$= 1 - \frac{\left(x^2 + \frac{\pi}{30}x + \frac{\pi^2}{3600}\right)}{2}$$ M1

$$= 1 - \frac{\pi^2}{7200} - \frac{\pi}{60}x - \frac{x^2}{2}$$ A1

b $\int_0^{0.5} \cos\left(x + \frac{\pi}{60}\right) dx = \left[\sin\left(x + \frac{\pi}{60}\right)\right]_0^{0.5}$ or use of GDC M1

= (0.52469....) − (0.05233...)

= 0.4724 A1

c $\int_0^{0.5} 1 - \frac{\pi^2}{7200} - \frac{\pi}{60}x - \frac{x^2}{2} dx$ M1

$$= \left[x - \frac{\pi^2}{7200}x - \frac{\pi}{120}x^2 - \frac{x^3}{6}\right]_0^{0.5}$$ M1

$= 0.47196799\ldots$ A1

$$\frac{0.4720 - 0.4724}{0.4724} \times 100$$ M1

$\approx -0.085\%$ A1

8 **a** $|z_1 z_2| = r_1 r_2$ (M1)

$= \sqrt{3^2 + 5^2} \times \sqrt{(-1)^2 + (4\sqrt{5})^2}$ M1

$= 9\sqrt{34}$ A1

b $\left|\frac{z_2}{z_1}\right| = \frac{r_2}{r_1}$ (M1)

$= \dfrac{9}{\sqrt{34}} = \dfrac{9\sqrt{34}}{34}$ M1 A1

c $(3+5\text{i})(3+5\text{i}) = 9+30\text{i}-25$ M1

$-16+30\text{i}$ A1

d Let $\sqrt{z_2} = z = a+bi$

$z = \sqrt{-1+4\sqrt{5}\text{i}} \Rightarrow (a+b\text{i})^2 = -1+4\sqrt{5}\text{i}$ M1

$a^2 + 2ab\text{i} - b^2 = -1+4\sqrt{5}\text{i}$ M1

$\left.\begin{array}{l} a^2 - b^2 = -1 \\ 2ab = 4\sqrt{5} \end{array}\right\}$ so $b = \dfrac{2\sqrt{5}}{a} \Rightarrow a^2 - \left(\dfrac{2\sqrt{5}}{a}\right)^2 = -1$ M1

$a^2 - \dfrac{20}{a^2} + 1 = 0 \Rightarrow a^4 + a^2 - 20 = 0$ M1

$(a^2+5)(a^2-4) = 0 \Rightarrow a^2 = 4 \Rightarrow a = \pm 2$ M1

[a is real so discard (a^2+5)] R1

$\Rightarrow b = \pm\sqrt{5}$ M1

$z = 2+\sqrt{5}\text{i}$ and $z = -2-\sqrt{5}\text{i}$ A1

9 $\overrightarrow{AB} = \begin{pmatrix} 4 \\ -5 \\ -6 \end{pmatrix}, \overrightarrow{AC} = \begin{pmatrix} 3 \\ 3 \\ -11 \end{pmatrix}, \overrightarrow{AD} = \begin{pmatrix} 3 \\ 5 \\ -4 \end{pmatrix}$ (M1)A1A1A1

$\overrightarrow{AB} \cdot \overrightarrow{AC} \times \overrightarrow{AD} = \begin{vmatrix} 4 & -5 & -6 \\ 3 & 3 & -11 \\ 3 & 5 & -4 \end{vmatrix}$ or equivalent attempted M1

$\overrightarrow{AB} \cdot \overrightarrow{AC} \times \overrightarrow{AD} = 241$ A1

Volume $= \dfrac{241}{6} \textit{units}^3$ A1

10 a Calculates two points on the line.

e.g. $P(4,-2,5)$ and $Q(5,4,8)$ (M1)A1A1

Finds two non-parallel direction vectors of the plane

e.g. $\overrightarrow{AP} = \begin{pmatrix} 1 \\ -4 \\ 0 \end{pmatrix}, \overrightarrow{AQ} = \begin{pmatrix} 2 \\ 2 \\ 3 \end{pmatrix}$ (M1)A1A1

Finds normal vector to the plane

$$\text{e.g. } \overrightarrow{AP} \times \overrightarrow{AQ} = \begin{pmatrix} -12 \\ -3 \\ 10 \end{pmatrix}$$ M1A1

Since vector equation of plane is $\mathbf{r.n} = \mathbf{a.n}$, finds $\mathbf{a.n}$ using the normal vector and either A, P or Q

$$\text{e.g. } \begin{pmatrix} -12 \\ -3 \\ 10 \end{pmatrix} \cdot \begin{pmatrix} 3 \\ 2 \\ 5 \end{pmatrix} = 8$$ M1A1

$-12x - 3y + 10z = 8$ or equivalent form A1

b Cross product of the normals to both planes will give direction vector of the line:

$$\text{e.g. } \begin{pmatrix} -12 \\ -3 \\ 10 \end{pmatrix} \times \begin{pmatrix} 33 \\ -28 \\ 9 \end{pmatrix} = \begin{pmatrix} 253 \\ 438 \\ 435 \end{pmatrix}$$ M1A1

$$\therefore \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 3 \\ 2 \\ 5 \end{pmatrix} + \mu \begin{pmatrix} 253 \\ 438 \\ 435 \end{pmatrix}$$ (use of point A) M1, correct line A1

c Use of $\cos\theta = \dfrac{\mathbf{n}_1 \cdot \mathbf{n}_2}{|\mathbf{n}_1||\mathbf{n}_2|}$ M1

$$\cos\theta = \left| \frac{\begin{pmatrix} -12 \\ -3 \\ 10 \end{pmatrix} \cdot \begin{pmatrix} 33 \\ -28 \\ 9 \end{pmatrix}}{\left|\begin{matrix} -12 \\ -3 \\ 10 \end{matrix}\right| \left|\begin{matrix} 33 \\ -28 \\ 9 \end{matrix}\right|} \right| = \frac{222}{\sqrt{253}\sqrt{1954}} \approx 0.315741$$ A1A1

(accept the negative value if Modulus not used)

$\theta = 71.6^{\circ}$ A1

11 a $$L: \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -1 \\ 4 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} 3 \\ -1 \\ 5 \end{pmatrix} = \begin{pmatrix} -1+3\lambda \\ 4-\lambda \\ 2+5\lambda \end{pmatrix}$$ M1

$$\begin{pmatrix} -1+3\lambda \\ 4-\lambda \\ 2+5\lambda \end{pmatrix} \cdot \begin{pmatrix} 3 \\ 1 \\ -2 \end{pmatrix} = 7$$ (M1)

$3(-1+3\lambda) + (4-\lambda) - 2(2+5\lambda) = 7$ M1

$\lambda = -5$ A1

$$P = \begin{pmatrix} -1+3(-5) \\ 4-(-5) \\ 2+5(-5) \end{pmatrix} = \begin{pmatrix} -16 \\ 9 \\ -23 \end{pmatrix}$$ (M1)A1

b **Step 1**: Finds shortest distance from *R* to *L.*

Finds a general point *X* on line *L* and the vector $\overrightarrow{RX}$

e.g. $\overrightarrow{OX} = \begin{pmatrix} -1+3\lambda \\ 4-\lambda \\ 2+5\lambda \end{pmatrix}, \overrightarrow{RX} = \begin{pmatrix} -1+3\lambda \\ 4-\lambda \\ 2+5\lambda \end{pmatrix} - \begin{pmatrix} -3 \\ -7 \\ 8 \end{pmatrix} = \begin{pmatrix} 2+3\lambda \\ 11-\lambda \\ -6+5\lambda \end{pmatrix}$ M1A1

Finds dot product of $\overrightarrow{RX}$ with direction vector of *L* its set equal to zero to make $\overrightarrow{RX}$ perpendicular to *L* (this is the shortest distance between *R* and *L).*

e.g. $\begin{pmatrix} 2+3\lambda \\ 11-\lambda \\ -6+5\lambda \end{pmatrix} \cdot \begin{pmatrix} 3 \\ -1 \\ 5 \end{pmatrix} = 0$ M1

$\lambda = 1$ A1

$\overrightarrow{OX} = \begin{pmatrix} 2 \\ 3 \\ 7 \end{pmatrix}, \overrightarrow{RX} = \begin{pmatrix} 5 \\ 10 \\ -1 \end{pmatrix}$ A1A1

Finds distance $|RX| = \sqrt{126} = 11.22...$ (M1)A1

Step 2: Finds shortest distance from R to Π.

Finds dot product of *R* with normal of Π

$\begin{pmatrix} 3 \\ 1 \\ -2 \end{pmatrix} \cdot \begin{pmatrix} -3 \\ -7 \\ 8 \end{pmatrix} = -32$ M1 A1

Calculates distance between plane constants

$\dfrac{7-(-32)}{\sqrt{3^2+1^2+(-2)^2}} = \dfrac{39}{\sqrt{14}} = 10.42.....$ M1A1

Step 3: By comparison, point R is closer to Π than to *L* R1

10 Equivalent systems of representation: more complex numbers

1 Find the values of the real parameter m so that the vectors $p = mi + 2j - 3k$ *and* $r = (2-m)i + mj - 7k$ are perpendicular.

2 Show that the vector $\mathbf{a} = \cos\left(\frac{3\pi}{4}\right)\mathbf{i} - \frac{\sqrt{5}}{\sqrt{20}}\mathbf{j} + \sin\left(\frac{11\pi}{6}\right)\mathbf{k}$ is a unit vector.

3 Prove that for any two non-zero vectors **a** and **b**, $\mathbf{a}\times\mathbf{b} + \mathbf{b}\times\mathbf{a} = \mathbf{0}$.

4 The triangle ABC, and the vectors **AB**, **BC** and **CA** are given.

a Show that **AB** + **BC** + **CA** = **0**.

b Use the relation in part a to show that $\mathbf{AB}\times\mathbf{BC} = \mathbf{BC}\times\mathbf{CA} = \mathbf{CA}\times\mathbf{AB}$.

c Given that the angle between the vectors **CA** and **AB** is denoted by θ, show that $\sin \text{BAC} = \sin\theta$.

d Hence use the vector product to show the sine rule.

5 The position vectors $\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3, \mathbf{a}_4$ and $\mathbf{a}_5$ represent the vertices A_1, A_2, A_3, A_4 and A_5 of a pentagon. The position vectors $\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3$ and $\mathbf{p}_4$ represent the midpoints P_1, P_2, P_3 and P_4 of the sides $[A_2A_3]$, $[A_3A_4]$, $[A_4A_5]$ and $[A_5A_1]$ respectively.

a Draw the diagram and find the vectors $\mathbf{P_1P_3}$ and $\mathbf{P_2P_4}$ in terms of the position vectors of the vertices.

b Given that the points M and N are the midpoints of sides $[P_1P_3]$ and $[P_2P_4]$ respectively, show that the line segment [MN] is parallel to the side $[A_1A_2]$.

c Determine the ratio A_1A_2:MN.

6 The vectors $\mathbf{AB} = \begin{pmatrix} 5 \\ -3 \\ 4 \end{pmatrix}$ and $\mathbf{AC} = \begin{pmatrix} -1 \\ 4 \\ 6 \end{pmatrix}$ are given.

a Find the angle between the vectors **AB** and **AC**.

b Hence or otherwise find the area of the triangle ABC, giving your answer correct to 2 decimal places.

7 The triangle with the vertices A(3,0), B(-1,4) and C(-2,1) is given in the coordinate plane.

a Find the vectors **AB**, **BC** and **CA**.

b Find the vectors **p, q** and **r** that have the same length and are perpendicular as vectors **AB**, **BC** and **CA** respectively, that are facing outside the triangle.

c Show that vectors **p**, **q** and **r** form a triangle.

Exam-style questions

8 Let $z = 1 + i$ and $w = \sqrt{3} + i$.

a *Find zw, giving your answer in the form $a + bi$, for $a, b \in \mathbb{R}$. (2)

b *Write

i z **ii** w

in the form $rcis\theta, r \in \mathbb{R}^+, 180^\circ < \theta \le 180^\circ$. (2)

c Hence, write zw in the form $rcis\theta, r \in \mathbb{R}^+, 180^\circ < \theta \le 180^\circ$. (2)

d Hence, find the exact values of

i $\cos 75^\circ$ **ii** $\sin 75^\circ$. (5)

9 A geometric progression has first term $a = 2$ and common ratio $r = 1 + i$.

By writing the common ratio in modulus-argument form, find

a the 9th term

b the sum of the first 8 terms.

Give your answers to parts **a** and **b** in Cartesian form. (8)

10 Let $w = 50 + 50\sqrt{3}i$.

a Express w in in the form $rcis\theta$, where $r \in \mathbb{R}^+$, and $-180^\circ < \theta \le 180^\circ$. (2)

b Hence, solve $z^2 = 50 + 50\sqrt{3}i$, giving your answers in the form $z = rcis\theta$, where $r \in \mathbb{R}^+$ and $-180^\circ < \theta \le 180^\circ$. (6)

c Write down the solutions to $z^2 = 50 + 50\sqrt{3}i$ in Cartesian form $a + bi$, where a and b are real numbers given in exact form. (2)

11 Let $w = -2 - 2i$.

a Write w in the modulus-argument form $rcis\theta$, where $r \in \mathbb{R}^+$ and $-180^\circ < \theta \le 180^\circ$. (2)

b Hence, solve $z^3 = -2 - 2i$, giving your answers in the form $z = rcis\theta$, where $r \in \mathbb{R}^+$ and $-180^\circ < \theta \le 180^\circ$. (7)

c Write the solution to part **b** which lies in the 4th quadrant of the argand diagram in the form $a + bi$. (1)

d Write an expression for the expansion of $(a + bi)^3$ using the binomial theorem. Give your answer in the form $A + Bi$. (2)

e Hence, find a solution to the simultaneous equations $\begin{cases} a^3 - 3ab^2 = -2 \\ 3a^2b - b^3 = -2 \end{cases}$ for which both a and b are integers. (3)

12 Let $z = 2e^{\frac{i\pi}{5}}$.

a **Find the following in the form $re^{i\theta}, r \in \mathbb{R}^+, -\pi < \theta \leq \pi$.

i z^2 **ii** z^6 **iii** z^* **iv** $\frac{1}{z}$ **v** $\frac{1}{z^3}$ (10)

b Find the smallest value of n, $n \in \mathbb{Z}^+$, so that z^n, is purely real. State what this value of z^n is. (3)

c Find the smallest value of m, $m \in \mathbb{Z}^+$, so that z^m, is purely real and positive. State what this value of z^m is. (3)

d Prove that it is impossible for z^k to be purely imaginary for $k \in \mathbb{Z}^+$. (3)

Answers

1 $m \in \{-3, 7\}$

2 **2** $|a| = \sqrt{\left(\cos\frac{3\pi}{4}\right)^2 + \left(\frac{\sqrt{5}}{\sqrt{20}}\right)^2 + \left(\sin\frac{11\pi}{6}\right)^2} = \sqrt{\left(\frac{1}{\sqrt{2}}\right)^2 + \frac{1}{4} + \left(\frac{1}{2}\right)^2} = 1$

3 $b \times a = -a \times b \Rightarrow a \times b + b \times a = 0$

4 **a** $AB = b - a,\ BC = c - b,\ CA = a - c \Rightarrow AB + BC + CA = b - a + c - b + c - a = 0$

b $AB \times BC = AB \times (-AB - CA) = -AB \times CA = CA \times AB$

$BC \times CA = BC \times (-AB - BC) = -BC \times AB = AB \times BC$

c Extend the vectors **CA** and **AB**, and the angle θ is the angle between the lines CA and AB.

d $CA \times AB = |CA||AB|\sin C\hat{A}B = BC \times CA = |BC||CA|\sin B\hat{C}A$, so $\frac{|AB|}{\sin B\hat{C}A} = \frac{|BC|}{\sin C\hat{A}B}$ and similarly $\frac{|AB|}{\sin B\hat{C}A} = \frac{|BC|}{\sin C\hat{A}B} = \frac{|CA|}{\sin A\hat{B}C}$.

5 **a** $P_1P_3 = \frac{1}{2}(a_5 + a_4 - a_3 - a_2)$, $P_2P_4 = \frac{1}{2}(a_1 + a_5 - a_4 - a_3)$

b $A_2A_1 = a_1 - a_2$,

$MN = N - M = \frac{1}{2}(P_4 + P_2 - P_1 - P_3) = \frac{1}{4}(a_1 + a_5 + a_4 + a_3 - a_2 - a_3 - a_4 - a_5) = \frac{1}{4}(a_1 - a_2)$

c 4:1

6 **a** 1.43 rad

b 25.5

7 **a** $-4i + 4j, -i - 3j, 5i - j$

b $4i + 4j, -3i + j, -i - 5j$

c $p + q + r = 0$

8 **a** $zw = (1 + \mathrm{i})(\sqrt{3} + \mathrm{i}) = \sqrt{3} - 1 + (\sqrt{3} + 1)\mathrm{i}$ M1A1

b **i** $z = \sqrt{2}cis45^\circ$ **ii** $w = 2cis30^\circ$ M1A1M1A1

c $zw = 2\sqrt{2}cis75^\circ$ M1A1

d **i** $2\sqrt{2}\cos 75^\circ = \mathrm{Re}(zw)$ R1

$2\sqrt{2}\cos 75^\circ = \sqrt{3} - 1$ by the result in part **a** M1

$\Rightarrow \cos 75^\circ = \dfrac{\sqrt{3}-1}{2\sqrt{2}}$ A1

ii $2\sqrt{2}\sin 75^\circ = \mathrm{Im}(zw)$

$2\sqrt{2}\sin 75^\circ = \sqrt{3}+1$ by part **a** M1

$\Rightarrow \sin 75^\circ = \dfrac{\sqrt{3}+1}{2\sqrt{2}}$ A1

9 $1+i = \sqrt{2}cis\dfrac{\pi}{4}$ A1A1

a $u_9 = 2\left(\sqrt{2}cis\dfrac{\pi}{4}\right)^8 = 2\times 2^4 cis2\pi = 32$ M1A1A1

b $S_8 = 2\dfrac{\left((1+i)^8-1\right)}{(1+i)-1} = -30i$ M1A1A1

10 a Modulus is $50\sqrt{1+3} = 100$, $\tan(\arg) = \sqrt{3} \Rightarrow \arg = 60^\circ$ A1A1

$w = 100cis60^\circ$

b $(rcis\theta)^2 = r^2 cis2\theta = 100cis60^\circ$ M1A1

$r = 10, 2\theta = 60 \pm 360n \Rightarrow \theta = 30 \text{ or } -150$ M1M1

$z = 10cis30 \text{ or } 10cis(-150)$ A1A1

c From part **b**, $z = 5\sqrt{3}+5i$ or $z = -5\sqrt{3}-5i$ A1A1

11 a $\sqrt{8}cis(-135^\circ)$ A1A1

b $(rcis\theta)^3 = r^3 cis3\theta = \sqrt{8}cis(-135^\circ)$ M1A1

$r = \sqrt{2}, 3\theta = -135 \pm 360n \Rightarrow \theta = -45 \text{ or } 75 \text{ or } -165$ M1M1

$z = \sqrt{2}cis(-45^\circ) \text{ or } \sqrt{2}cis75^\circ \text{ or } \sqrt{2}cis(-165^\circ)$ A1A1A1

c $z = \sqrt{2}cis(-45) = 1-i$ A1

d $a^3 + 3a^2bi - 3ab^2 - b^3i = (a^3 - 3ab^2) + (3a^2b - b^3)i$ M1A1

e using previous parts, after equating coefficients $a = 1$ $b = -1$ R1A1A1

12 a i $z^2 = 4e^{i\frac{2\pi}{5}}$ M1A1

ii $z^6 = 2^6 e^{i\frac{6\pi}{5}} = 64e^{i\frac{-4\pi}{5}}$ M1A1

iii $z^* = 2e^{i\frac{-\pi}{5}}$ M1A1

iv $\frac{1}{z} = \frac{1}{2}e^{i\frac{-\pi}{5}}$ M1A1

v $\frac{1}{z^3} = \frac{1}{8}e^{i\frac{-3\pi}{5}}$ M1A1

b $z^n = 2^n e^{\frac{in\pi}{5}}$ for this to be purely real $\frac{n\pi}{5}$ must be a multiple of π R1

So smallest value of n is 5 . Value is $2^5 e^{i\pi} = -32$. A1A1

c $z^m = 2^m e^{\frac{im\pi}{5}}$ for this to be purely real and positive $\frac{m\pi}{5}$ must be a multiple of 2π R1

So smallest value of m is 10 . Value is $2^{10} e^{i2\pi} = 1024$. A1A1

d $z^k = 2^k e^{\frac{ik\pi}{5}}$ for this to be purely imaginary $\frac{k\pi}{5}$ must be an odd multiple of $\frac{\pi}{2}$ R1

This implies that $2k = 5 \times$ (an odd integer) which is impossible. A1R1

11 Valid comparisons and informed decisions: probability distributions

1 A continuous random variable *X* has PDF function given by

$$f(x)=\begin{cases} a\sec^2 x & \text{for } 0\le x\le \frac{\pi}{4} \\ 0 & \text{otherwise} \end{cases}$$

a State the value of *a.*

b Find the median value of *X.*

c Calculate $\mathrm{P}\left(X\le \frac{\pi}{6}\right)$.

2 A continuous random variable X has a probability function

$$f(x)=\begin{cases} Ax(4-x)^2 & 0\le x\le 4 \\ 0 & \text{otherwise} \end{cases}$$

a find the value of A

b calculate the mean, mode, variance and standard deviation

3 A continuous random variable X has the probability density function

$$f(x)=\frac{k}{x(4-x)},1\le x\le 3$$

Use partial fractions to find k and calculate the exact value of the mode, mean and variance.

4 The probability distribution of a discrete random variable X is given by

$P(X=r)=kr, r=1,2,3,\ldots n$ where k is a constant.

Find the value of k.

5 In a group of 6 students, 4 are male and 2 are female, determine how many committees of 3 students can be chosen containing one male and two female.

6 A continuous random variable X has a probability distribution function given by $f(x)=k(1+\cos x), 0\le x\le \pi$, f(x)=0 otherwise.

a Given that k is a constant, find the value of k

b Find correct to 4 decimal places the mean, μ

8 A continuous probability density function is given by $f(x) = ax - bx^2$, $0 \le x \le 2$. It is found that E(X)=1.

a Write down 2 simultaneous equations for *a* and *b*.

b Find the values of *a* and *b*.

c Find the variance of the distribution.

d If two independent observations are made on X, find the probability that at least one of them is less than a half.

Exam-style questions

9 Consider the strictly increasing function defined by $f(x) = \dfrac{2x}{x^2+1}, x \ge 0$.

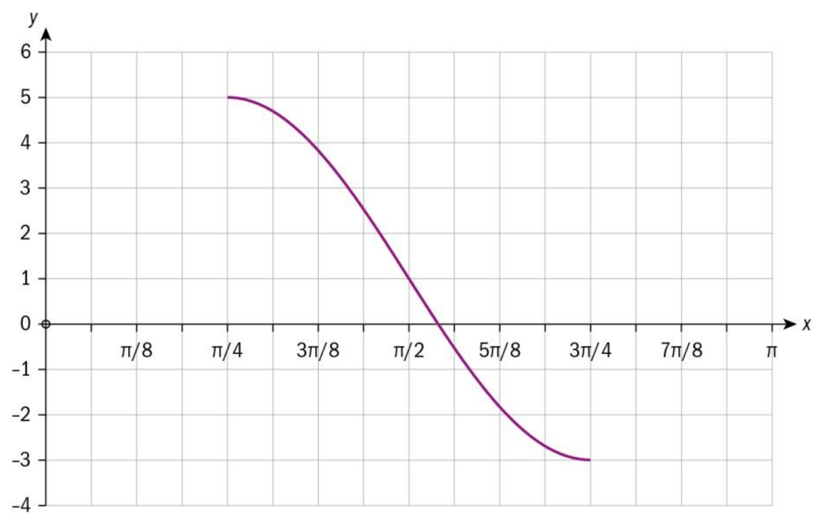
a Show that $f'(x) = \dfrac{2(1-x^2)}{(x^2+1)^2}, x \ge 0$. (4)

b Explain why the range of *f* is the interval [0,1]. (5)

c Find $\int \dfrac{2x}{x^2+1} dx$. (2)

d Hence find the area enclosed by the graph of f and the x-axis between 0 and 1. (3)

10 The diagram shows part of the graph of $f(x) = A + B\sin(Cx)$ for real constants *A*, *B* and *C*, and *x* in radians.



a State the range of the function. (1)

b Using your answer to part **a**, justify that *A*=1. (1)

c Given that *B*>0, show that *B*=4. (2)

d Justify that the function has period π . (2)

e Hence, find the value of *C* and justify your answer. (1)

Let $g(x) = \sin x$.

f The graph of g can be transformed onto the graph of f. Describe the sequence of transformations required to obtain the graph of f from the graph of g, stating clearly the order in which they must be performed. (3)

11 **a** The exponential distribution with parameter λ is given by the probability distribution function $f(x) = Ae^{-\lambda x}$, $0 \le x < \infty$. Find A. (2)

b The cumulative distribution function is defined as $F(x) = \mathrm{P}(X \le x)$. Calculate $F(x)$ for the function $f(x) = Ae^{-\lambda x}$, $0 \le x < \infty$ in part **a**. (2)

c Show that X satisfies $\mathrm{P}(X \ge x + z \mid X \ge z) = \mathrm{P}(X \ge x)$. (3)

d Find $\mathrm{E}(X)$ and $\mathrm{var}(X)$. (5)

e Find the median and mode of X. (3)

f Suppose that X_1 and X_2 are independent exponentially distributed random variables with parameters λ and μ respectively. Show that $Y = \min(X_1, X_2)$ is also exponentially distributed and find its parameter. (3)

12 A fair m-sided dice, with sides numbered 1 to m, is rolled n times.

a Find the expected number of times a one is obtained. (3)

b Let A_k be the event that the largest roll is less than or equal to a value k. By considering the sample space and the set of successful outcomes, find $\mathrm{P}(A_k)$. (2)

c Hence, explain why the probability that k is the largest value obtained is $\dfrac{k^n - (k-1)^n}{m^n}$. (2)

d Find the probability that the sum of all the dice throws is greater than $n+1$. (3)

13 The Poisson distribution is a distribution which can be used to model the number of occurrences of an event in some specified time. The Poisson distribution with parameter (or rate) λ is given by $\mathrm{P}(X = k) = \dfrac{\lambda^k}{k!}e^{-\lambda}$, $k = 0, 1, 2, 3, \ldots$.

a Dan is active on Twitter and fires off tweets at an average rate of 4 per hour.

By modelling the number of tweets Dan makes in 1 hour as a Poisson variable, find the probability that in a given hour he tweets

i 4 times **ii** more than once. (3)

b Let X and Y be independent Poisson random variables, both with parameter λ. Show that

i $\mathrm{P}(X + Y = k) = \dfrac{(2\lambda)^k}{k!}e^{-2\lambda}$ **ii** $\mathrm{P}(X = k \mid X + Y = n) = \dbinom{n}{k}\left(\dfrac{1}{2}\right)^n$ (5)

c Deduce the distributions of $X + Y$ and $X \mid X + Y = n$. (2)

Answers

1 **a** 1 **b** $\arctan\frac{1}{2}$ **c** $\frac{\sqrt{3}}{3}$

2 **a** A=3/64

b mean=1.6, mode =2

$E(X^2)=3.2$-(1.6)(1.6)= 0.64

variance = 0.64, standard deviation = 0.8

3 $k=\frac{2}{\ln 3}$, mode=2, mean = 2, variance = $4-\frac{4}{\ln 3}$

4 $k=\frac{2}{n(n+1)}$

5 4

6 **a** $k=\frac{1}{\pi}$ **b** 0.9342

8 **a** $\int_0^2 \left(ax-bx^2\right)dx=1$

$2a-\frac{8}{3}b=1$ and

$\int_0^2 x(ax-bx^2)dx=1$

$\frac{8}{3}a-4b=1$

b a=1.5, b=0.75

c 0.2

d 0.288

9 **a** $f'(x)=\frac{(2x)'(x^2+1)-2x(x^2+1)'}{(x^2+1)^2}=\frac{2(x^2+1)-2x\cdot 2x}{(x^2+1)^2}$ M1A1A1

$=\frac{2-2x^2}{(x^2+1)^2}=\frac{2(1-x^2)}{(x^2+1)^2}$ A1

b $f(0)=0$ and $\lim\limits_{x\to\infty} f(x)=0$ R1

$f'(x)=0 \Rightarrow x=1$ A1

Since $f(x)>0$ for $x>0$, so f has a maximum at $x=1$ and $f(1)=1$ A1

As f is strictly increasing, $f'(x)>0 \Rightarrow x<1$ R1

Therefore, the range of f is the interval [0,1] AG

c By inspection, $\int \frac{2x}{x^2+1}dx = \ln(x^2+1)+C$ M1A1

d $\int_0^1 \frac{2x}{x^2+1}dx = \left[\ln(x^2+1)\right]_0^1 = \ln 2$ M1A1A1

10 a $-3 \le y \le 5$ A1

b $A = \frac{5+(-3)}{2} = 1$ R1AG

c $|B| = \frac{5-(-3)}{2} = 4$ M1

As $B>0$, so $B=4$ R1AG

d The difference between the x-coordinates of the maximum and minimum is $\frac{3\pi}{4} - \frac{\pi}{4} = \frac{\pi}{2}$ .M1

Therefore the period is $2 \times \frac{\pi}{2} = \pi$ R1AG

e $C = \frac{2\pi}{\text{period of } f}$ M1

So $C = \frac{2\pi}{\pi} = 2$ A1

f Vertical stretch by factor 4 and horizontal stretch by factor $\frac{1}{2}$ (in any order) A1A1

Followed by a vertical translation of 1 unit upwards A1

11 a $\int_0^\infty f(x)dx = A\int_0^\infty e^{-\lambda x}dx = A\left[-\frac{e^{-\lambda x}}{\lambda}\right]_0^\infty = \frac{A}{\lambda} = 1 \Rightarrow A = \lambda$ M1A1

b $F(x) = P(X \le x) = \int_0^x f(t)dt = \left[-e^{-\lambda t}\right]_0^x = 1 - e^{-\lambda x}$ M1A1

c $P(X \ge x) = 1 - P(X \le x) = e^{-\lambda x}$ M1

$$P(X \ge x+z \mid X \ge z) = \frac{P((X \ge x+z) \cup (X \ge z))}{P(X \ge z)} = \frac{P(X \ge x+z)}{P(X \ge z)} = \frac{e^{-\lambda(x+z)}}{e^{-\lambda z}} = e^{-\lambda x} = P(X \ge x)$$

M1A1

d Integrating by parts with $u = x, v' = \lambda e^{-\lambda x}, u' = 1, v = -e^{-\lambda x}$,

$E(X) = \int_0^\infty \lambda x e^{-\lambda x}dx = [-xe^{-\lambda x}]_0^\infty + \int_0^\infty e^{-\lambda x}dx = 0 + \frac{1}{\lambda} = \frac{1}{\lambda}$. M1A1

(Understanding that $\lim_{x\to\infty}(e^{-\lambda x}) \to 0$ quicker than $x \to \infty$, so $\lim_{x\to\infty}(-xe^{-\lambda x}) = 0$ R1)

Similarly, integrating by parts with $u = x^2, v' = \lambda e^{-\lambda x}, u' = 2x, v = -e^{-\lambda x}$, ,

$$E(X^2) = \int_0^\infty \lambda x^2 e^{-\lambda x} dx = [-2xe^{-\lambda x}]_0^\infty + \frac{2}{\lambda}\int_0^\infty \lambda x e^{-\lambda x} dx = 0 + \frac{2}{\lambda^2} = \frac{2}{\lambda^2}$$

and $\text{var}(X) = E(X^2) - E(X)^2 = \frac{2}{\lambda^2} - \frac{1}{\lambda^2} = \frac{1}{\lambda^2}$. M1A1

e $f(x) = \lambda e^{-\lambda x}$ has a maximum at $x = 0$ in $0 \le x < \infty$ so the mode is 0. A1

$P(X \le m) = \int_0^m f(t)dt = 1 - e^{-\lambda m} = \frac{1}{2} \Rightarrow e^{-\lambda m} = \frac{1}{2} \Rightarrow m = \frac{1}{\lambda}\ln 2$ M1A1

f $P(Y \ge t) = P(\min(X_1, X_2) \ge t) = P((X_1 \ge t) \cup (X_2 \ge t))$.

As X_1 and X_2 are independent, $P((X_1 \ge t) \cup (X_2 \ge t)) = P(X_1 \ge t)P(X_2 \ge t)$ R1

and $P(Y \ge t) = e^{-\lambda t}e^{-\mu t} = e^{-(\lambda+\mu)t}$. So Y is exponential with parameter $\lambda + \mu$. M1A1

12 a Let X be the number of ones obtained. $X \sim B\left(n, \frac{1}{m}\right)$. R1

$E(X) = np = \frac{n}{m}$. M1A1

b For each roll, there are m possibilities, so there are a total of m^n combinations of n dice rolls, or $n(U) = m^n$. Similarly, there are k possibilities to obtain less than or equal to k on each dice roll and so $n(A_k) = k^n$. Hence, $P(A_k) = \frac{k^n}{m^n}$. M1A1

c The probability that k is the largest value obtained is equal to the probability that the largest roll is less than or equal to k minus the probability that the largest roll is less than k , or $P(A_k) - P(A_{k-1}) = \frac{k^n - (k-1)^n}{m^n}$. R1AG

d Let Y be the sum of the n dice throws. $\min(Y) = n$ as each roll has minimum value 1.

$P(Y > n+1) = 1 - P(Y \le n+1) = 1 - P(Y = n) - P(Y = n+1)$ (M1)

For $Y = n$, all the dice rolls must have been equal to 1 and for $Y = n+1$, $n-1$ of the dice rolls must have been equal to 1 with the other dice roll equal to 2. The probability of obtaining a given value is $\frac{1}{m}$, independently of other dice rolls. Therefore

$P(Y = n) = \left(\frac{1}{m}\right)^n$ and $P(Y = n+1) = \binom{n}{1}\left(\frac{1}{m}\right)^{n-1}\left(\frac{1}{m}\right) = n\left(\frac{1}{m}\right)^n$. (M1)

$P(Y > n+1) = 1 - P(Y \le n+1) = 1 - (1+n)\left(\frac{1}{m}\right)^n$. (A1)

13 a Let X be the number of tweets Dan sends in an hour, which is Poisson with rate 4.

i $P(X = 4) = \frac{4^4}{4!}e^{-4} = \frac{32}{3}e^{-4} = 0.19537$ (5 d.p.) (A1)

ii $P(X > 1) = 1 - P(X = 0) - P(X = 1) = 1 - \frac{4^0}{0!}e^{-4} - \frac{4^1}{1!}e^{-4} = 1 - e^{-4} - 4e^{-4} = 0.90842$ (5 d.p.) (M1,A1)

b i $P(X+Y=k)=\sum_{l=0}^{k}P(X=1)P(Y=k-1)=\sum_{l=0}^{k}\left(\frac{\lambda^l}{l!}e^{-\lambda}\right)\left(\frac{\lambda^{k-l}}{(k-l)!}e^{-\lambda}\right)$

$P(X+Y=k)=\lambda^k e^{-\lambda}\sum_{l=0}^{k}\frac{1}{l!(k-l)!}=\frac{\lambda^k e^{-\lambda}}{k!}\sum_{l=0}^{k}\frac{1}{l!(k-l)!}=\frac{\lambda^k e^{-\lambda}}{k!}\sum_{l=0}^{k}\binom{k}{l}$ (M1)

$(1+x)^k=\sum_{l=0}^{k}\binom{k}{l}x^k, x=1\Rightarrow\sum_{l=0}^{k}\binom{k}{l}=2^k$ and $P(X+Y=k)=\frac{(2\lambda)^k}{k!}e^{-k}$ (M1,A1)

ii $P(X=k\mid X+Y=n)=\frac{P((X=k)\cup(X+Y=n))}{P(X+Y=n)}=\frac{P(X=k,X+Y=n)}{P(X+Y=n)}$

$=\frac{\left(\frac{\lambda^k}{k!}e^{-k}\right)\left(\frac{\lambda^{n-k}}{(n-k)!}e^{-\lambda}\right)}{\frac{(2\lambda)^n}{n!}e^{-2\lambda}}=\frac{\frac{\lambda^n}{k!(n-k)!}e^{-2\lambda}}{\frac{(2\lambda)^n}{n!}e^{-2\lambda}}=\binom{n}{k}\left(\frac{1}{2}\right)^n$ M1A1

c The distribution of $X+Y$ is Poisson with parameter 2λ and the distribution of $X\mid X+Y=n$ is binomial with parameters n and $\frac{1}{2}$. R2